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Fibrations and Congruence Towers of Arithmetic Hyperbolic Manifolds

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Alla mia famiglia: a chi è appena arrivato,
a chi c'è da sempre,
a chi non c'è più.

Contents

Introduction

From the 80's of the last century onwards, the interest on hyperbolic geometry has grown very much and has been the object of study in many different aspects. William P. Thurston's notes "*The Geometry and Topology of 3-Manifolds*" [Thu80] introduced several new ideas into Geometric Topology and more in general all of his work was fundamental in hyperbolic geometry.

In 1982, Thurston wrote an article [Thu82] which ended with a list of 24 open questions concerning 3-manifolds and Kleinian groups that he found fascinating. Among them there was the famous Thurston's geometrization conjecture, proved by Perelman in early 2000's, and the so-called virtual fibering conjecture, which we report here:

18. *Does every hyperbolic 3-manifold have a finite-sheeted cover which fibers over the circle? This dubious-sounding question seems to have a definite chance for a positive answer.*

During the years, many of the 24 Thurston's question were answered, but the virtual fibering conjecture remained unanswered for 25 years and very few steps were made towards its solution. The situation changed mainly thanks to the work of Ian Agol [Ago08] and Daniel T. Wise, and the question has been finally answered affirmatively. Nowadays, only the 23th question, which talks about volumes of hyperbolic 3-manifolds, remains unanswered, and researchers know almost nothing in that direction.

Agol's work is based on the property of a group Γ being virtually RFRS, that is to say virtually *residually finite rationally solvable.* In [Ago08], the author proved that certain hyperbolic 3-manifolds with a virtually RFRS fundamental group virtually fiber. Eventually, this was used in order to prove that every hyperbolic 3-manifold is virtually fibered [AGM12]. However, the proof is not constructive, hence finding a cover which effectively fibers remains an open problem.

In this thesis, we follow an article by Ian Agol and Matthew Stover [AS19] and describe a criterion for an arithmetic hyperbolic lattice to admit a RFRS tower consisting entirely of congruence subgroups. We primarily use this to prove that:

Theorem 0.1. *The Bianchi groups* $\mathrm{PSL}(2, \mathcal{O}_d)$ *with* $d \not\equiv -1 \pmod 8$ *and* d *positive square-free contain a RFRS tower consisting entirely of congruence subgroups. In particular, these Bianchi orbifolds virtually fiber on a congruence cover.*

The method also works in greater generality. Let k be a number field, $\mathcal{O}_k$ its ring of integers and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_k$ with residue characteristic p. If we take a suitable algebraic group $\mathcal{G}$ over k and consider a finite index subgroup $\Gamma \leq \Gamma(\mathfrak{p})$, where $\Gamma(\mathfrak{p})$ is the principal congruence subgroup of level $\mathfrak{p}$ in the arithmetic lattice $\mathcal{G}_{\mathcal{O}_k}$, we can define a RFRS tower for Γ whenever Γ^{ab} has no p-torsion. The construction is explicit and is done in the following way.

Using the fact that $\mathcal{G}_k$ commensurates Γ, we can find a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{G}_k$ such that the sequence of subgroups $\{\Gamma_n\}_{n\in\mathbb{N}}$, where

$$\Gamma_n = \bigcap_{j=0}^{n} g_j \Gamma g_j^{-1},$$

is a RFRS tower for Γ. The key will be to find an initial $g_1 \in \mathcal{G}_k$ so that

$$\Gamma/(\Gamma \cap g_1 \Gamma g_1^{-1})$$

is an elementary abelian p-group. Then, inductively, we will find g_n using a particular simplicial complex called Bruhat-Tits building associated to $\mathcal{G}_{k_\mathfrak{p}}$, where $k_\mathfrak{p}$ is the completion of k with respect to the $\mathfrak{p}$-adic valuation.

The thesis is organized as follows: in the first chapter we will recall basic notions of hyperbolic geometry and we will present some geometric background about the virtual fibration problem; the second chapter will treat algebraic notions and preliminaries about number fields, non-Archimedean valuations and buildings; finally, in the third chapter we will present the main result of this work, following [AS19].

Chapter 1

Hyperbolic manifolds and the Fibration problem

This first chapter aims to recall basic constructions of hyperbolic geometry, starting with the definition of the hyperbolic n-dimensional space $\mathbb{H}^n$ and of hyperbolic manifolds, and then moving on to the more specified setting of 3-manifolds. Finally, we will present the problem of fibering and virtual fibering, citing some results which will help to understand the context of our main results.

Most of the results in this chapter can be found in many introductory texts about differential geometry or hyperbolic geometry, such as [BP92], [Mar20] or [Mar16].

1.1 Models for Hyperbolic Space

Let $n \geq 2$. This section is devoted to the construction of the n-dimensional hyperbolic space $\mathbb{H}^n$. More precisely, we will define some *models* for $\mathbb{H}^n$, that is to say Riemaniann n-manifolds which are isometric one to each other. To avoid confusion and emphasize the specific representation of the manifold, different symbols will be used for them: I^n, D^n or H^n. The symbol $\mathbb{H}^n$ will be used to denote any Riemannian n-manifold isometric to the hyperbolic space.

1.1.1 Hyperboloid Model

The n-sphere $\mathbb{S}^n$ is the set of all norm-1 points in $\mathbb{R}^{n+1}$ considered with the standard scalar product. In the same fashion, if we consider a non-positive definite scalar product on $\mathbb{R}^{n+1}$ we could also consider the set of points with norm equal to -1. We define:

Definition 1.1. Let $\langle \cdot, \cdot \rangle$ be the Lorentzian scalar product of signature $(n, 1)$ on $\mathbb{R}^{n+1}$, i.e. defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

where x_i and y_i are the components of the vectors $x, y \in \mathbb{R}^{n+1}$. A vector x is said to be *time-like*, *light-like* or *space-like* is $\langle x, x\rangle$ is respectively negative, null or positive.

We define the *hyperboloid model* I^n as

$$I^n := \left\{x \in \mathbb{R}^{n+1} \;\middle|\; \langle x, x\rangle = -1,\ x_{n+1} > 0\right\}.$$

Observe that I^n is one of the connected components of the two-sheeted hyperboloid $\{x \mid \langle x, x\rangle = -1\}$. The 2-dimensional case is shown in Figure 1.1, which suggests that this might not be the best representation of $\mathbb{H}^n$ from the viewpoint of visualization of the space. I^3 would live inside $\mathbb{R}^4$, and hence we should find some other model in order to graphically represent it better.

However, we will see that the hyperboloid I^n is instead a very useful model to work with geodesics and isometries.

First of all, we prove that:

Proposition 1.2. *I^n is a Riemannian n-manifold.*

Proof. Consider the map $f\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, defined by $f(x) = \langle x, x\rangle$. It is a smooth map and its differential at any point $x \in \mathbb{R}^{n+1}$ is $df_x(y) = 2\langle x, y\rangle$. Since it is surjective for every $x \neq 0$, it follows that the value -1 is a regular value for f and hence that $f^{-1}(-1)$ is a codimension-1 differential submanifold inside $\mathbb{R}^{n+1}$. This proves in particular that I^n is an n-manifold.

Again, from the definition of the differential df_x we note that the tangent space $T_x I^n$ at every $x \in I^n$ is $x^\perp$: indeed it holds that

$$T_x I^n = \operatorname{Ker} df_x = \left\{y \in \mathbb{R}^{n+1} \;\middle|\; \langle x, y\rangle = 0\right\} = x^\perp.$$

Now, since x is time-like, its orthogonal space endowed with the restriction of the Lorentzian scalar product is positive defined and it defines a metric tensor on I^n. □

We say that the I^n is the hyperboloid model for the hyperbolic space $\mathbb{H}^n$.

In this model, it is very easy to classify the isometries of $\mathbb{H}^n$. Let $\mathrm{O}(n, 1; \mathbb{R})$ be the group of linear isomorphisms of $\mathbb{R}^{n+1}$ which preserve the Lorentzian scalar product, i.e. the set of $F \in \mathrm{GL}(n+1, \mathbb{R})$ such that $\langle F(v), F(w)\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^{n+1}$. It is obvious that an element in $\mathrm{O}(n, 1; \mathbb{R})$ preserves the two-sheeted hyperboloid, and that elements preserving the upper sheet I^n form an index-2 subgroup of $\mathrm{O}(n, 1; \mathbb{R})$, which will be called $\mathrm{O}^+(n, 1; \mathbb{R})$.

Moreover, we can prove that:

Proposition 1.3. *The group of isometries* $\mathrm{Isom}(I^n)$ *of the hyperboloid I^n is equal to* $\mathrm{O}^+(n, 1; \mathbb{R})$.

Proof. Isometries of the ambient manifold $\mathbb{R}^{n+1}$ which preserve I^n also map orthogonal complements to orthogonal complements isometrically, hence $\mathrm{O}^+(n, 1; \mathbb{R}) \subseteq \mathrm{Isom}(\mathbb{H}^n)$.

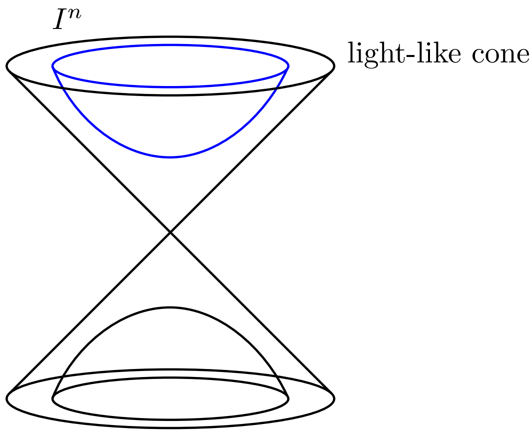

Figure 1.1: The two-sheeted hyperboloid in $\mathbb{R}^3$. The upper sheet in blue is I^n, while the cone represented is the set of isotropic vector (light-like vectors).

For the other inclusion, we recall that isometries of Riemannian manifolds are very rigid, in the sense that an isometry is fully determined by its first-order behaviour at a point x.

Having this in mind, we need to prove that given any $x, y \in I^n$ and any linear isometry $g \colon x^\perp \longrightarrow y^\perp$, there exists an $F \in \mathrm{O}^+(n, 1; \mathbb{R})$ such that $F(x) = y$ and $F|_{x^\perp} = g$. But simple calculations show that $\mathrm{O}^+(n, 1; \mathbb{R})$ acts transitively on I^n and hence we can suppose without loss of generality that $x = y = (0, \ldots, 0, 1)$. Under this assumption, $x^\perp$ is the horizontal hyperplane and $g \in \mathrm{O}(n; \mathbb{R})$ is an $n \times n$ orthogonal matrix in the Euclidean sense. Defining F as

$$F := g \oplus (1) = \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right)$$

we conclude the other inclusion. □

Usually a good understanding of a Riemannian manifold comes from its geodesics, and for I^n things are not different.

Definition 1.4. A *line* in I^n is the intersection of a 2-dimensional subspace W of $\mathbb{R}^{n+1}$ with I^n, when it is not empty.

More generally, a *k-subspace* in I^n is the intersection of a $(k+1)$-dimensional subspace W of $\mathbb{R}^{n+1}$ with I^n, when it is not empty. The $(n-1)$-subspaces in I^n are called *hyperplanes*.

Along with subspaces, we can consider the *reflection* r_S with respect to the subspace S, which is defined as follows: by definition of a subspace, $S = I^n \cap W$ and we can decompose $\mathbb{R}^{n+1} = W \perp W^\perp$; we then set $r_S|_W = id_W$ and $r_S|_{W^\perp} = -id_{W^\perp}$. Reflections help us to show that:

Proposition 1.5. *Geodesics in I^n are precisely the lines run at constant speed. Concretely, if $p \in I^n$ is any point and $v \in T_pI^n$ is a unit vector, the geodesic γ exiting from p with velocity v is*

$$\gamma(t) = \cosh(t) \cdot p + \sinh(t) \cdot v.$$

As a consequence, the space I^n is complete.

Proof. Let W be the plane in $\mathbb{R}^{n+1}$ spanned by p and v. The line determined by W contains p and is tangent to v, and using the reflection r_l we see that the geodesic γ is contained in the fixed locus of r_l. Indeed, r_l fixes p and v and hence the geodesic γ is preserved too.

The curve $\alpha(t) = \cosh(t) \cdot p + \sinh(t) \cdot v$ parametrizes the whole line l and its velocity is unitary for every t. This completes the proof that $\alpha = \gamma$.

For what concerns the completeness of I^n, it is sufficient to note that the geodesics we just parametrized are globally defined, hence the space is geodesically complete and complete by the Hopf-Rinow Theorem. □

Reflections can also help to describe the isometries in the models we will present hereafter, where there won't be a precise characterization like the one of Proposition 1.3:

Proposition 1.6. *Reflections along hyperplanes generate* $\mathrm{Isom}(I^n)$.

Proof. It is a standard linear algebra result that the group $\mathrm{O}(n;\mathbb{R})$ is generated by reflections along hyperplanes. Thus we have that the stabilizer of any point in I^n is generated by reflections. Since reflections act transitively on points in $\mathbb{R}^{n+1}$, we have concluded the proof. □

1.1.2 The Poincaré Disk Model

As said before, the hyperboloid model cannot capture well every aspect of the hyperbolic space. That's the reason why we define the *Poincaré disk* model D^n. As a set it is nothing more than the usual n-dimensional disk

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\},$$

with a specific metric tensor which makes it isometric to the previous defined model, I^n.

Let us now construct a particular diffeomorphism p between I^n and D^n from which the metric tensor on D^n is induced. Consider $\mathbb{R}^{n+1}$ with the same Lorentzian scalar product as before, and we identify $\mathbb{R}^n$ with the horizontal hyperplane $\{x_{n+1} = 0\}$. We define $p(x)$ for $x \in I^n$ any point on the hyperboloid to be the intersection of the segment which connects x and $P = (0, \ldots, 0, -1)$ with the horizontal hyperplane $\mathbb{R}^n$. In Figure 1.2 we can see the construction. It is also easy to write p explicitly: we have that

$$p(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{x_{n+1}+1}, \ldots, \frac{x_n}{x_{n+1}+1}\right).$$

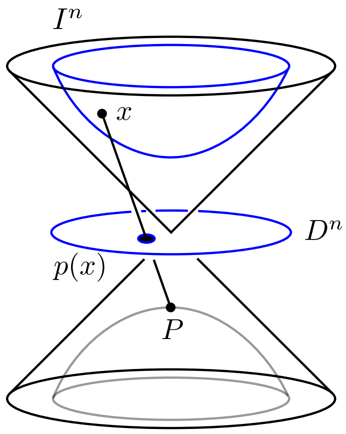

Figure 1.2: The construction of the bijection $p\colon I^n \longrightarrow D^n$ induced by the projection on $P = (0, \dots, 0, -1)$.

Using this map we define the metric tensor g on D^n as the pullback metric with respect to the map p^{-1}. Explicitly we have:

Proposition 1.7. *The metric tensor g at a point $x \in D^n$ is*

$$g_x = \left(\frac{2}{1-\|x\|^2}\right)^2 \cdot g^E_x,$$

where $\|x\|$ denotes the Euclidean norm of the vector $x \in D^n$ and g^E denotes the Euclidean metric tensor on $D^n \subseteq \mathbb{R}^n$.

Proof. To use the definition of the pullback metric we write the inverse function p^{-1} in coordinates as follows:

$$p^{-1}(x) = \left(\frac{2x_1}{1-\|x\|^2}, \dots, \frac{2x_n}{1-\|x\|^2}, \frac{1+\|x\|^2}{1-\|x\|^2}\right).$$

We then should calculate the differential of p^{-1} at a generic point $x \in D^n$, but if we observe that rotations in $\mathbb{R}^{n+1}$ along the x_{n+1}-axis preserve both I^n and D^n and are isometries for both of them, we can suppose without loss of generality that the point x is of the form $x = (x_1, 0, \dots, 0)$. Now it is easier to compute the differential:

$$dp_x^{-1} = \frac{2}{1-x_1^2} \cdot \begin{pmatrix} \frac{1+x_1^2}{x_1^2} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \frac{2x_1}{1-x_1^2} & 0 & \dots & 0 \end{pmatrix}.$$

Since the column vectors of the matrix form an orthonormal basis of the tangent space $T_{p^{-1}(x)}I^n$, it holds that dp_x^{-1} stretches all vector by the constant $\frac{2}{1-x_1^2}$, and the thesis follows. $\square$

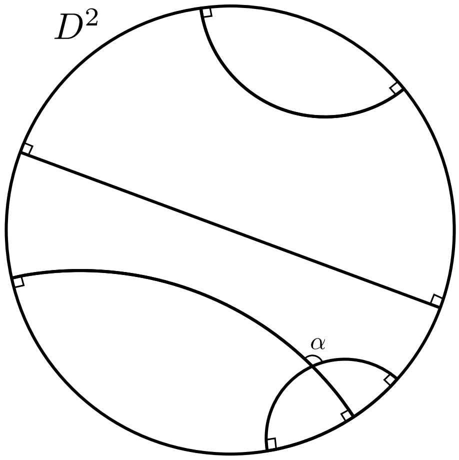

Figure 1.3: The Poincaré disk model for $\mathbb{H}^2$. Geodesics are circles and lines orthogonal to ∂D^n, and angles correspond to Euclidean angles.

The previous proposition tells us that the Poincaré disk model can be very useful in representing the hyperbolic space $\mathbb{H}^n$ when it is important to work with angles between vectors: indeed, if a metric g is such that $g_x = \lambda(x) g_x^E$ for a smooth positive function λ, then g is called a *conformal* metric, and though lengths are distorted with respect to the Euclidean ones, angles are instead preserved.

Having characterized the geodesics in the hyperboloid model I^n as the lines in I^n run at constant speed, we can again use the isometry p to understand how they can be seen in the Poincaré disk model. It turns out, doing explicit calculations that we omit here, that:

Proposition 1.8. *The geodesics in the Poincaré disk model (D^n, g) are straight lines and portions of circles orthogonal to ∂D^n run at constant speed.*

More generally, k-subspaces in D^n are the intersections of D^n with k-planes and k-spheres orthogonal to ∂D^n.

In Figure 1.3 we can see the 2-dimensional hyperbolic space $\mathbb{H}^2$ with the Poincaré disk model. Every curve drawn is a geodesic with respect to the hyperbolic metric g.

Having already written explicitly geodesics in the hyperboloid model, it is really easy to write them in the Poincaré disk model at least in the case where the geodesics exit from the origin. Using the isometry p we obtain:

Proposition 1.9. *The geodesic γ in D^n exiting from the origin with unit velocity $v \in S^{n-1}$ is*

$$\gamma(t) = \frac{\sinh(t)}{\cosh(t)+1} \cdot v = \tanh\left(\frac{t}{2}\right) \cdot v.$$

Hence the ball in D^n centered in the origin with radius r, i.e. the set of all points at distance less than r from the origin in D^n, is an Euclidean ball with radius $\tanh(r/2)$.

The last statement of the previous proposition helps also us to obtain information about the *sectional curvature* of the hyperbolic space.

Recall that in a Riemannian manifold (M, g), for every point $p \in M$ there exists an $\varepsilon > 0$ such that the ball $B_p(\varepsilon)$ centered in p with radius ε is diffeomorphic to an Euclidean ball. The g-volume of this ball $B_p(\varepsilon)$ will be usually different to the volume of an Euclidean ball with the same radius. The general notion of *curvature* of a Riemannian manifold encodes discrepancies between these values.

Definition 1.10. Let (M, g) be a surface. The *gaussian curvature* K_p at a point $p \in M$ is defined as

$$K_p := \lim_{\varepsilon \to 0} \left(\left(\pi\varepsilon^2 - \mathrm{Vol}_g(B_p(\varepsilon)) \right) \cdot \frac{12}{\pi\varepsilon^4} \right).$$

In other words, the following formula holds:

$$\mathrm{Vol}_g(B_p(\varepsilon)) = \pi\varepsilon^2 - \frac{\pi\varepsilon^4}{12} K + o(\varepsilon^4)$$

where $o(\varepsilon^4)$ is the Landau-symbol.

The reason to the $\frac{\pi}{12}$ normalization is that it makes the unit sphere S^2 to have gaussian curvature equal to 1. If the Riemannian manifold (M, G) has dimension $n \geq 3$ we define the *sectional curvatures* to be:

Definition 1.11. Let (M, G) be a Riemannian manifold. Let $p \in M$ be a point and let S be a smooth surface passing through p inside M with the Riemannian structure induced by g. The sectional curvature of (M, g) along (p, S) is the gaussian curvature of S in p.

Actually, it turns out that the sectional curvature does not depend on the choice of the surface S, but only on the choice of the tangent plane of S at p. Hence the sectional curvature is defined on pairs (p, W) where $p \in M$ is a point and $W \subseteq T_pM$ is a 2-dimensional subspace in the tangent space at p.

In our setting, we have proved that the hyperbolic space has the highest number of symmetries, that is to say the group $\mathrm{Isom}(\mathbb{H}^n)$ acts transitively on points and *frames*, i.e. orthonormal basis of $T_p\mathbb{H}^n$. This implies that the sectional curvatures all coincide, and hence it just suffices to calculate the sectional curvature in a particularly simple case.

Let us work in the Poincaré disk model, and consider the intersection $D^n \cap \{x_3 = \ldots = x_n = 0\}$, which coincides with the Poincaré disk model D^2 of the hyperbolic plane $\mathbb{H}^2$. Let $p = 0$ and let us calculate the gaussian curvature of D^2 at p. We have:

Proposition 1.12. *The area $A(r)$ of the ball $B_0(r)$ centered in 0 and with radius r inside D^2 is*

$$A(r) = 4\pi \sinh^2\left(\frac{r}{2}\right) = 2\pi(\cosh(r) - 1).$$

Proof. It is an easy calculus, knowing that the ball $B_0(r)$ in D^2 with respect to the hyperbolic metric is exactly the Euclidean ball centered in 0 and with radius $\tanh(r/2)$. □

Corollary 1.13. *The hyperbolic space* $\mathbb{H}^n$ *has constant sectional curvature* $K = -1$.

Proof. It is sufficient to use the Taylor expansion of the hyperbolic tangent, obtaining that

$$A(r) = 2\pi\left(\cosh(r) - 1\right) = 2\pi\left(\frac{r^2}{2!} + \frac{r^4}{4!} + o(r^4)\right) = \pi r^2 + \frac{\pi r^4}{12} + o(r^4)$$

and hence $K = -1$. □

1.1.3 The Half-Space Model

The last model we are going to present is the half-space model, which is again a conformal model for the hyperbolic space $\mathbb{H}^n$, and which will be very useful later on in order to characterize in a different way the group of isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$.

It is obtained from the Poincaré disk model through a diffeomorphism from the punctured Euclidean space $\mathbb{R}^n \setminus \{(0, \dots, 0, -1)\}$ to itself called *inversion*.

Definition 1.14. Let $S = S(x_0, r)$ be the sphere in $\mathbb{R}^n$ with radius r and centered in x_0. The *inversion* along S is the diffeomorphism $\varphi_S \colon \mathbb{R}_n \setminus \{x_0\} \longrightarrow \mathbb{R}^n \setminus \{x_0\}$ defined as

$$\varphi_S(x) = x_0 + r^2 \frac{x - x_0}{\|x - x_0\|^2}.$$

An inversion along a sphere S switches the inside with the outside of S, and a visual description of an inversion can be seen in Figure 1.4.

Similar arguments to those used in the proof of Proposition 1.7 show that:

Proposition 1.15. *The inversion* φ_S *along a sphere* S *is a smooth* anticonformal *map, i.e. its differential* $d(\varphi_S)_x$ *at any point* x *is the product of a positive scalar* $\lambda(x)$ *and an orientation-reversing isometry. Moreover,* φ_S *maps* k*-spheres and* k*-planes to* k*-spheres and* k*-planes (not necessarily spheres to spheres or planes to planes, though).*

Proof. Without loss of generality, up to translations we can suppose that the centre of the sphere $S = S(x_0, r)$ is the origin $x_0 = 0$. Then $\varphi_S(x) = r^2 \frac{x}{\|x\|^2}$ for any $x \in \mathbb{R}^n \setminus \{0\}$ and we have to calculate the differential $d(\varphi_S)_x$ at a point x. Up to rotations centered in 0 we can also suppose $x = (x_1, 0, \dots, 0)$, and calculating the partial derivatives at x we find that

$$d(\varphi_S)_x = \frac{r^2}{x_1^2} \cdot \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

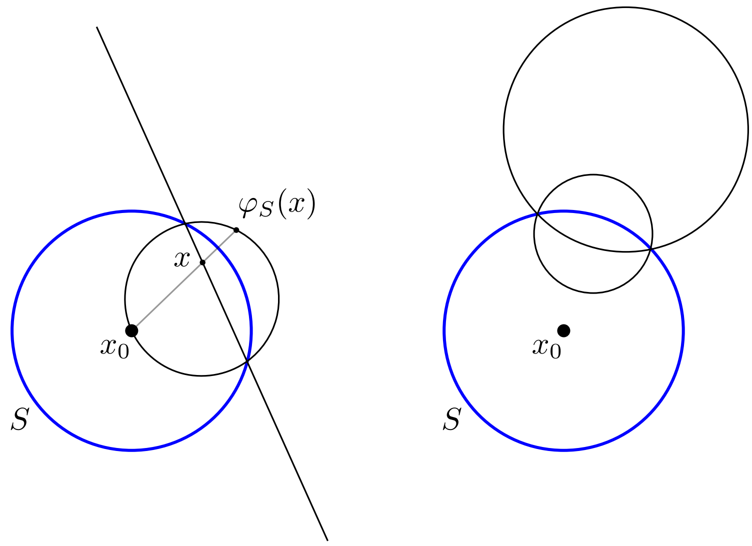

Figure 1.4: The inversion φ_S along the blue sphere S. Circles passing through x_0 transform to lines, while circles disjoint from x_0 transform to circles.

that is to say $d(\varphi_S)_x$ is the product of the positive scalar $\frac{r^2}{\|x\|^2}$ and the reflection with respect to the hyperplane orthogonal to x.

The fact that φ_S preserves the family of k-spheres and k-planes is a classical result in Euclidean geometry. $\square$

If we identify S^n with $\mathbb{R}^n \cup \{\infty\}$ through the stereographic projection, we can extend the inversion φ_S to a self-diffeomorphism of the sphere S^n by setting $\varphi_S(x_0) = \infty$ and $\varphi_S(\infty) = x_0$. In this setting, it makes no sense to differentiate planes from spheres, as planes are just spheres passing through the point ∞. Hence using this extension of φ_S we can restate the second part of the previous proposition by saying that φ_S preserves k-spheres for every k.

The reason why we defined inversions is, as we already mentioned, to describe the half-space model H^n. Let us define it as a Riemannian manifold: it is the set

$$H^n := \{x \in \mathbb{R}^n \mid x_n > 0\}$$

with metric tensor induced by the diffeomorphism $\varphi_S \colon D^n \longrightarrow H^n$ where φ_S is the inversion along the sphere $S = S((0, \dots, 0, -1), \sqrt{2})$ centered in $(0, \dots, 0, -1)$ and with radius $\sqrt{2}$, as shown in Figure 1.5.

In order to obtain a bijection on the boundaries too, we define the boundary ∂H^n of the half-space model to be the horizontal hyperplane $\{x_n = 0\}$ together with a point ∞ at infinity, which is the image of the point $(0, \dots, 0, -1)$ under the extended inversion φ_S.

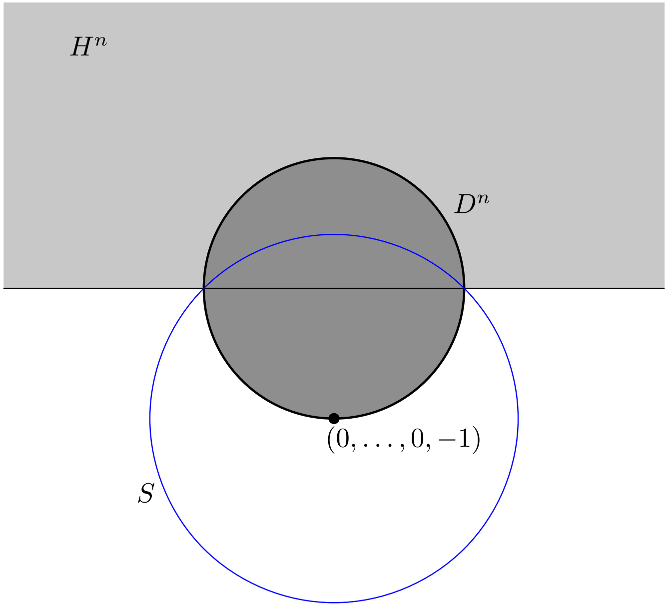

Figure 1.5: The half-space model H^n is the image of D^n with respect to the inversion along the blue sphere, which has radius $\sqrt{2}$ and is centered in $(0, \dots, 0, -1)$.

Since we proved that inversions are anticonformal and preserve k-spheres and k-planes, we have that:

Proposition 1.16. *The half-space model H^n is a conformal model of the hyperbolic space $\mathbb{H}^n$. Its geodesics are lines and circles orthogonal to ∂H^n, that is to say vertical lines and half-circles orthogonal to the horizontal hyperplane $\{x_n = 0\}$.*

More generally, k-subspaces are k-planes or k-spheres in $\mathbb{R}^n$ orthogonal to the horizontal hyperplane $\{x_n = 0\}$.

Doing some easy calculations for the differential of the inversion φ_S, it is easy to find the metric tensor on H^n which makes it isometric to D^n:

Proposition 1.17. *The metric tensor on H^n at a point x is*

$$g_x = \frac{1}{x_n^2} g_x^E,$$

where as usual g^E is the Euclidean metric tensor on $H^n \subseteq \mathbb{R}^n$.

Hence, this model is a conformal model, like the Poincaré disk, but its metric tensor is easier, being every affine horizontal hyperplane in H^n a Euclidean space with a scaled metric. Moreover, some isometries are easy to identify:

Proposition 1.18. *The following maps are isometries of H^n:*

- *horizontal affine transformations* $(x_1, \dots, x_n) \longmapsto (A(x_1, \dots, x_{n-1}) + b, x_n)$, *where* $A \in \mathrm{O}(n-1; \mathbb{R})$ *and* $b \in \mathbb{R}^{n-1}$*;*
- *homotheties* $x \longmapsto \lambda x$, *where* $\lambda > 0$*;*
- *inversions with respect to spheres orthogonal to* ∂H^n*.*

Proof. Horizontal affine transformations obviously preserve the metric tensor since the last coordinate is fixed and on the others the transformation is a Euclidean isometry.

In order to prove that homotheties and inversions are isometries we prove that they preserve the norm of tangent vectors $v \in T_x H^n$: we denote with $\| \cdot \|$ the hyperbolic norm and with $\| \cdot \|^E$ the Euclidean norm.

If $\varphi(x) = \lambda x$, we have

$$\|d\varphi_x(v)\| = \frac{\|d\varphi_x(v)\|^E}{\varphi(x)_n} = \frac{\lambda \|v\|^E}{\lambda x_n} = \frac{\|v\|^E}{x_n} = \|v\|.$$

If $\varphi(x)$ is an inversion, without loss of generality, up to composing with translations and homotheties, we can suppose that the sphere is centered in the origin and has unitary radius. Hence $\varphi(x) = \frac{x}{\|x\|^2}$. We already proved in Proposition 1.15 that $d\varphi_x(v) = \frac{1}{\|x^2\|} r(v)$ where r is a linear reflection. Therefore

$$\|d\varphi_x(v)\| = \frac{\|d\varphi_x(v)\|^E}{\varphi(x)_n} = \frac{\|v\|^E}{\|x\|^2 \varphi(x)_n} = \frac{\|v\|^E}{x_n} = \|v\|.$$

$\square$

Recall that the hyperplanes in the half-space model are half-spheres orthogonal to the horizontal boundary and vertical hyperplanes. Thus, reflections with respect to hyperplanes are represented in H^n by inversions along spheres and Euclidean reflections with respect to vertical hyperplanes.

We can then rephrase Proposition 1.6 in this context by stating:

Proposition 1.19. *The group* $\mathrm{Isom}(H^n)$ *is generated by inversions along spheres and reflections along Euclidean hyperplanes orthogonal to* ∂H^n*.*

1.2 Compactification of $\mathbb{H}^n$

It is useful to introduce a formal definition of the boundary $\partial \mathbb{H}^n$ of the hyperbolic space, which does not depend on the choice of the model, and to endow the space $\overline{\mathbb{H}}^n = \mathbb{H}^n \cup \partial \mathbb{H}^n$ with a topology compatible with the one of $\mathbb{H}^n$.

Points on the boundary are crucial especially in dimension 3, where our study will focus from the next section. Indeed, we will be able to identify isometries of $\mathbb{H}^3$ with 2×2 matrices thanks to their action on the boundary $\partial \mathbb{H}^3$.

Definition 1.20. Let a *geodesic half-line* in $\mathbb{H}^n$ be a geodesic $\gamma\colon [0,+\infty) \longrightarrow \mathbb{H}^n$ run at constant unit speed.

The set of the *points at infinity* $\partial\mathbb{H}^n$ in $\mathbb{H}^n$ is the set of geodesic half-lines modulo the following equivalence relation: $\gamma_1 \sim \gamma_2$ if and only if

$$\sup_{t\in[0,+\infty)} d(\gamma_1(t),\gamma_2(t)) < +\infty.$$

This definition turns out to be coherent, as it should be, to the definition of (topological) boundary in the Poincaré disk model D^n. In other words, there is a one-to-one correspondence between points of $\partial\mathbb{H}^n$ and points of ∂D^n. In this way we can also identify $\overline{\mathbb{H}}^n$ with $\overline{D}^n$ and provide the compactified hyperbolic space with the topology of $\overline{D}^n$. We could also define a topology on $\overline{\mathbb{H}}^n$ in an intrinsic way using the geodesic half-lines, but it is equivalent to the one of the Poincaré disk and we only mention it here for completeness.

Note that the compact space $\overline{\mathbb{H}}^n$ is nothing more than a topological space: points on the boundary have infinite distance from every other point in $\overline{\mathbb{H}}^n$.

However, isometries behave well when we consider the compactified hyperbolic space:

Proposition 1.21. *Every isometry $\varphi\colon \mathbb{H}^n \longrightarrow \mathbb{H}^n$ extends to a unique homeomorphism $\varphi\colon \overline{\mathbb{H}}^n \longrightarrow \overline{\mathbb{H}}^n$. Furthermore, an isometry φ is uniquely determined by its trace $\varphi|_{\partial\mathbb{H}^n}$ at the boundary.*

Proof. The extension of an isometry φ is natural: given any class $[\gamma]$ of geodesic half-lines, which is a boundary point, we define $\varphi([\gamma]) := [\varphi(\gamma)]$. The intrinsic definition of the topology on $\overline{\mathbb{H}}^n$ makes the extension a homeomorphism.

The fact that the trace at the boundary determines the isometry is proved by showing that if φ is an isometry such that $\varphi|_{\partial\mathbb{H}^n} = id$, then $\varphi = id$. Indeed, we just need the basic property that any point in $\mathbb{H}^n$ is the intersection of two lines. Since φ fixes the endpoints of every line (and hence every line as a set), it must fix the intersection of two lines. □

The last argument in the previous proof suggests that isometries can be divided into three families, depending on the fixed points on $\overline{\mathbb{H}}^n$:

Proposition 1.22. *Let $\varphi \neq id$ be an isometry of $\mathbb{H}^n$. Then one of the followings holds:*

- *φ has at least one fixed point in $\mathbb{H}^n$;*
- *φ has no fixed points in $\mathbb{H}^n$ and has exactly one fixed point in $\partial\mathbb{H}^n$;*
- *φ has no fixed points in $\mathbb{H}^n$ and has exactly two fixed points in $\partial\mathbb{H}^n$.*

In the first case φ is called elliptic*, in the second* parabolic *and in the latter* hyperbolic*.*

Proof. We need to prove that if φ fixes three different points p_1, p_2, p_3 on the boundary $\partial\mathbb{H}^n$, then it also fixes at least one point in $\mathbb{H}^n$. This point is characterized as the intersection of the line l with endpoints p_1 and p_2 with the unique line with endpoint p_3 and orthogonal to l. The isometry must fix both lines and hence their intersection. $\square$

1.3 Isometries of $\mathbb{H}^3$

From now on our work will be focused on the rich theory of 3-manifolds. We already characterized the group of isometries $\mathrm{Isom}(\mathbb{H}^n)$ using the hyperboloid model, for which the isometries are represented by 4×4 real matrices preserving the Lorentzian scalar product and the upper sheet of the hyperboloid. But if we use instead the half-space model and Proposition 1.21, it turns out that it can be possible to represent the isometries with 2×2 complex matrices.

Let $S = \mathbb{C} \cup \{\infty\}$ be the *Riemann sphere* upon which we consider the complex structure given by $\mathbb{C}$, and which is homeomorphic to S^2. Consider also the projective (special) linear group

$$\mathrm{PSL}(2,\mathbb{C}) = \mathrm{PGL}(2,\mathbb{C}) = \mathrm{GL}(2,\mathbb{C})/\{\lambda\,\mathrm{Id}\} = \mathrm{SL}(2,\mathbb{C})/\pm\mathrm{Id},$$

of invertible matrices (or equivalently of determinant-1 matrices), up to invertible complex scalars. These groups are equal since $\mathbb{C}$ is algebraically closed and hence any invertible matrix M can be written as

$$M = \sqrt{\det(M)}\left(\frac{1}{\sqrt{\det(M)}} \cdot M\right) = \sqrt{\det(M)} \cdot N,$$

where N belongs to $\mathrm{SL}(2,\mathbb{C})$.

Definition 1.23. A *Möbius transformation* is a map $S \longrightarrow S$ such that

$$z \longmapsto \frac{az+b}{cz+d},$$

where a, b, c, d are some complex numbers such that $ad - bc \neq 0$. It is an orientation-preserving self-diffeomorphism of S.

Hence any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$ determines a Möbius transformation $z \longmapsto \frac{az+b}{cz+d}$. The same matrix also determines:

Definition 1.24. A *Möbius anti-transformation* is a map $S \longrightarrow S$ such that

$$z \longmapsto \frac{a\overline{z}+b}{c\overline{z}+d},$$

where a, b, c, d are some complex numbers such that $ad - bc \neq 0$. It is an orientation-reversing self-diffeomorphism of S.

Some easy calculations show that the composition of two anti-transformations is a Möbius transformation and that transformations and anti-transformations together form a group.

Definition 1.25. The group $\mathrm{Conf}(S)$ of conformal maps is the set of all Möbius transformations and anti-transformations.

Notice that $\mathrm{Conf}(S) \triangleright \mathrm{PSL}(2,\mathbb{C})$ as an index-2 subgroup consisting of the Möbius transformations.

The next proposition is fundamental in order to link these diffeomorphisms to isometries of $\mathbb{H}^3$:

Proposition 1.26. *Inversions along circles and reflections with respect to lines are both Möbius anti-transformations and generate* $\mathrm{Conf}(S)$*.*

Proof. Observe first that both translations $z \longmapsto z+b$ and homotheties $z \longmapsto az$ (with $a \in \mathbb{C}^*$) are Möbius transformations, hence up to conjugating with some of them we obtain that every line reflection can be turned into the map $z \longmapsto \overline{z}$, which is clearly an anti-transformation. For what it regards circle inversions, again up to conjugating it is sufficient to prove that the inversion along the unit circle centered in 0 is an anti-transformation. But since this inversion is $z \longmapsto \frac{1}{\overline{z}}$, it is an anti-transformation too.

The proof that these anti-transformations generate the whole group $\mathrm{Conf}(S)$ can be done by observing that $\mathrm{Conf}(S)$ acts freely and transitively on triples of distinct points in S, and that the subgroup generated by the inversions and reflections act transitively on triples, too. □

Now, if we consider the half-space model H^3 and we identify $\mathbb{R}^3 = \mathbb{C}\times\mathbb{R} = \{(z,t) \mid z \in \mathbb{C}, t \in \mathbb{R}\}$, we see that the boundary $\partial H^3 = \{t = 0\} \cup \{\infty\}$ can be identified with the Riemann sphere

$$\partial H^3 = (\mathbb{C} \times \{0\}) \cup \{\infty\} = \mathbb{C} \cup \{\infty\} = S.$$

Thanks to Proposition 1.21 we then have the identification:

Proposition 1.27. *In the half-space model, identifying the boundary sphere with the Riemann sphere we have that*

$$\mathrm{Isom}(H^3) = \mathrm{Conf}(S).$$

Proof. Both groups are generated by the same set of elements, that is to say inversions along circles and reflections with respect to lines. □

As an immediate consequence, we have:

Corollary 1.28.

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C}).$$

In the next section we will define hyperbolic manifolds and we will see that they are determined by certain classes of subgroups of the isometry group. Hence it is important to know the isometry group of $\mathbb{H}^3$ and its properties.

The $\mathrm{PSL}(2,\mathbb{C})$-model for the isometries of $\mathbb{H}^3$ will come into play later on in the next chapters, when we will define arithmetic groups and arithmetic hyperbolic manifolds, which are the main ingredient for the final construction of this work.

1.4 Hyperbolic Manifolds

Hyperbolic manifolds are Riemannian manifolds which locally resemble the hyperbolic space. Formally:

Definition 1.29. A *hyperbolic n-manifold* (M, g) is a Riemannian manifold covered by open sets isometric to open sets of $\mathbb{H}^n$, i.e. for every $p \in M$ there exist open sets $U \subseteq M$ and $V \subseteq \mathbb{H}^n$ such that $p \in U$ and there is an isometry $\varphi\colon U \longrightarrow V$.

As an immediate consequence, hyperbolic manifolds have constant sectional curvature -1.

In this definition, the hyperbolic space $\mathbb{H}^n$ is used as a universal model for this family of manifolds. And the hyperbolic space is really unique, since it holds that:

Theorem 1.30. *Any complete simply connected hyperbolic manifold is isometric to* $\mathbb{H}^n$*.*

The previous theorem is fundamental in the study of complete hyperbolic manifolds with arbitrary fundamental group. Indeed, let $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ be a group of isometries: if it acts freely and properly discontinuously on $\mathbb{H}^n$, then the quotient $\mathbb{H}^n/\Gamma$ has a natural structure of smooth manifold and moreover it has a Riemannian structure coherent to the covering

$$\pi\colon \mathbb{H}^n \longrightarrow \mathbb{H}^n/\Gamma,$$

that is to say a metric that makes π a local isometry. Furthermore, being $\mathbb{H}^n$ complete, the quotient $\mathbb{H}^n/\Gamma$ is also a complete hyperbolic manifold, and its fundamental group is isomorphic to Γ. Observe also that $\mathbb{H}^n/\Gamma$ is orientable if and only if $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$.

But the converse is also true. Indeed:

Proposition 1.31. *A complete hyperbolic n-manifold is isometric to* $\mathbb{H}^n/\Gamma$ *for some* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *acting freely and properly discontinuously.*

Proof. Let M be a complete hyperbolic n-manifold. Its universal cover is a complete, simply connected, hyperbolic manifold: hence by the previous theorem it is isometric to $\mathbb{H}^n$. Now, the set Γ of deck transformations of the covering $\mathbb{H}^n \longrightarrow M$ is necessarily made of isometries. We can conclude then that $M = \mathbb{H}^n/\Gamma$ and Γ acts freely and properly discontinuously because the quotient is Hausdorff. $\square$

Recalling that an element in $\mathrm{Isom}(\mathbb{H}^n)$ with fixed points in $\mathbb{H}^n$ is called elliptic, and the basic result that Γ acts properly discontinuously if and only if it is discrete in $\mathrm{Isom}(\mathbb{H}^n)$ endowed with its compact-open topology, we have the following characterization of complete hyperbolic n-manifolds:

Corollary 1.32. *There is a natural one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{complete hyperbolic} \\ \text{manifolds up to isometry} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{discrete subgroups of } \mathrm{Isom}(\mathbb{H}^n) \\ \text{without elliptics} \\ \text{up to conjugation} \end{array} \right\}.$$

The description of this correspondence can be further modified by replacing the term "elliptic" with "torsion element". Indeed, in the hyperbolic space it makes sense to define the *barycenter* of a finite set of points, given in the hyperboloid model by the normalized of the usual barycenter defined in $\mathbb{R}^{n+1}$. From this it follows that a finite order isometry φ fixes the barycenter of the finite set of points $\{x, \varphi(x), \varphi^2(x), \ldots\}$ where x is any hyperbolic point. Conversely, since discrete groups of isometries act properly discontinuously the stabilizer of a fixed point of an elliptic isometry φ must be finite, and hence φ must have finite order. Thus we have proved that:

Proposition 1.33. *A discrete subgroup* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *is torsion-free if and only if it does not contain elliptic isometries.*

In general it is quite hard to produce explicit subgroups $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ which are discrete and without torsion, but if we relax the constraints and we only look for discrete subgroups, then the situation gets better.

Example 1.34. The subgroup $\mathrm{PSL}(2, \mathbb{Z}[i]) < \mathrm{PSL}(2, \mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ is clearly a discrete subgroup. However, it contains the order-2 element $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ which acts as the Möbius transformation $z \longmapsto -z$ on the Riemann sphere, and hence it acts as the π-rotation along the t-axis in $\mathbb{H}^3 = H^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$.

In the specific setting of $\mathbb{H}^3$, we recall that in the half-space model we have $\mathrm{Isom}^+(H^3) = \mathrm{PSL}(2, \mathbb{C})$. The following notation is commonly used:

Definition 1.35. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

Hence the group $\mathrm{PSL}(2, \mathbb{Z}[i])$ of the previous example is a Kleinian group. Many other examples of Kleinian groups will naturally come in mind in the next chapter, where we will give some background of algebraic number theory such as number fields and rings of integers, which generalize in some way the relation between $\mathbb{Q}$ and $\mathbb{Z}$. In the same way $\mathbb{Z}$ is used in the previous example in order to produce a discrete subgroup of matrices in $\mathrm{PSL}(2, \mathbb{C})$, we will be able to produce discrete subgroups by restricting the entries of matrices to subrings of $\mathbb{C}$, and we will call these subgroups *arithmetic subgroups.*

Given this difficulty in defining explicitly discrete subgroups without torsion, it is sometimes useful to focus on the study of hyperbolic manifolds up to a weaker relation than being isometrically diffeomorphic.

Definition 1.36. Two subgroups $H_1, H_2 < G$ are said to be *commensurable* if $H_1 \cap H_2$ is a finite index subgroup of both H_1 and H_2. They are *commensurable in the wide sense* if there exists a $g \in G$ such that H_1 and gH_2g^{-1} are commensurable.

We then define:

Definition 1.37. Two complete hyperbolic manifolds $M_1 = \mathbb{H}^n/\Gamma_1$ and $M_2 = \mathbb{H}^n/\Gamma_2$ are said to be *commensurable* if Γ_1 and Γ_2 are commensurable in the wide sense in $\mathrm{Isom}(\mathbb{H}^n)$. In this case, there exists a complete hyperbolic manifold M such that M covers both M_1 and M_2 and the covering degrees are both finite.

The last definition can be generalized to quotients $\mathbb{H}^n/\Gamma$ with Γ a discrete group of isometries. In this setting, we aren't working with manifolds anymore, but with *orbifolds*. We don't define these objects since we won't use them explicitly in this work; it is sufficient to think at orbifolds as manifolds with some singular points.

Similarly, Riemannian coverings generalize to orbifold coverings coherently.

The reason why we weaken the isometry-equivalence relation is based on the following important algebraic result:

Lemma 1.38 (Selberg)**.** *Every finitely generated subgroup* $\Gamma < \mathrm{GL}(n, \mathbb{C})$ *is virtually torsion-free, i.e. there exists a finite index subgroup* $\Gamma' < \Gamma$ *without torsion.*

Hence, from this we obtain that every discrete group $\Gamma < O^+(n, 1; \mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^n)$ is commensurable with a torsion-free discrete subgroup Γ', and that the hyperbolic manifold $\mathbb{H}^n/\Gamma'$ lies in the same commensurability class as the hyperbolic orbifold $\mathbb{H}^n/\Gamma$.

1.5 The Virtual Fibration Problem

The commensurability equivalence relation turns out to be useful to work with when dealing about properties which are *virtual*.

Definition 1.39. Given a property P, a group G is said to be *virtually* P if there exists a finite-index subgroup $H \leq G$ which satisfies P.

In a similar fashion, we define:

Definition 1.40. Given a property P, a hyperbolic manifold M is said to be *virtually* P if there exists a finite-degree cover $M' \twoheadrightarrow M$ such that M' satisfies P.

We have already seen, for example, that Kleinian groups are virtually torsion-free, thanks to the Selberg's Lemma. In order to define another property P which motivates the entire thesis we recall the definition of a fibration of manifolds:

Definition 1.41. Let F, E and B be smooth manifolds. A *fibration* (or *fiber bundle*) over the *base space* B with *fiber* F is a map

$$\pi \colon E \longrightarrow B$$

where E is called the *total space* such that it is locally trivial, i.e. for every point $p \in B$ there is an open neighbourhood $U \subseteq B$ of p whose pre-image $\pi^{-1}(U)$ is diffeomorphic to the product $U \times F$ via a map $\varphi \colon \pi^{-1}(U) \longrightarrow U \times F$ satisfying $\pi_1(\varphi(q)) = \pi(q)$ where $\pi_1 \colon U \times F \longrightarrow U$ is the projection onto the first factor.

We will be interested in fibrations over the circle S^1 with total space a hyperbolic 3-manifold, and hence with fiber F which is a surface. Such fibrations are also called *surface bundles* over the circle.

Every fibration over the circle can be described in a quite explicit way, thanks to the following result:

Theorem 1.42. *Any fibration of smooth manifolds $E \longrightarrow B$ with fiber F over a contractible base space is trivial, i.e. we have $E \cong F \times B$.*

If we remove a point p from S^1, we obtain a contractible space since $S^1 \setminus \{p\}$ is diffeomorphic to an open interval. As a corollary, we then have that:

Corollary 1.43. *Every surface bundle over the circle is a* mapping torus*, i.e. any fibration $\Sigma \hookrightarrow M \longrightarrow\!\!\!\!\!\rightarrow S^1$ with fiber Σ a surface is obtained from a homeomorphism $\phi\colon \Sigma \longrightarrow \Sigma$ in the following way:*

$$M \cong T_\phi := (\Sigma \times [0,1]) \,/\, \{(x,0) \sim (\phi(x),1)\}\,.$$

A fibration over the circle $\Sigma \hookrightarrow M \longrightarrow\!\!\!\!\!\rightarrow S^1$ always induces an exact sequence at the level of fundamental groups:

$$0 \longrightarrow \pi_1(\Sigma) \hookrightarrow \pi_1(M) \twoheadrightarrow \pi_1(S^1) = \mathbb{Z} \longrightarrow 0\ .$$

In general, if M is a compact manifold then Σ is also compact. At the level of fundamental groups, this implies that $\pi_1(\Sigma)$ is finitely generated and hence the following property holds:

Proposition 1.44. *Let $p\colon M \longrightarrow S^1$ be a surface bundle over the circle with M a compact 3-manifold. Then the induced map $p_*\colon \pi_1(M) \longrightarrow \pi_1(S^1) = \mathbb{Z}$ is an* algebraic fibration*, i.e. it is a surjection with finitely generated kernel.*

Now, a converse of this fact is also true, and it is due to Stallings:

Theorem 1.45 ([Sta62])**.** *Let M be an irreducible compact hyperbolic 3-manifold. If $\pi_1(M)$ has a finitely generated normal subgroup G, whose quotient group is $\mathbb{Z}$, then G is the fundamental group of an embedded surface Σ in M and there exists a fibration $\Sigma \hookrightarrow M \twoheadrightarrow S^1$.*

In 1982, Thurston wrote an article [Thu82] which ended with a list of 24 open questions concerning 3-manifolds and Kleinian groups that he found fascinating. Among them there was the famous Thurston's geometrization conjecture, proved by Perelman in early 2000's, and the so-called *virtual fibering conjecture*, which we report here:

18. *Does every hyperbolic 3-manifold have a finite-sheeted cover which fibers over the circle? This dubious-sounding question seems to have a definite chance for a positive answer.*

The origin of this question comes directly from the other aforementioned conjecture, which asked if any oriented closed 3-manifold could be decomposed into geometric pieces, i.e. into 3-manifolds admitting a finite-volume geometric structure. Indeed, in another article (unpublished, see [Thu98]) Thurston proved that the geometrization conjecture holds for manifolds which fiber over the circle, and hence an affirmative answer to the virtual fibering conjecture would imply the geometrization one.

Although during the years many of the 24 Thurston's question were answered, the virtual fibering conjecture remained unanswered for 25 years and very few steps were made towards its solution. The situation changed mainly thanks to the work of Ian Agol [Ago08] and Daniel T. Wise, and the question has been finally answered affirmatively.

Agol's work is based on the property of a group Γ being virtually RFRS, that is to say virtually *residually finite rationally solvable.* We will return on this definition in its full details only in the third chapter, after having talked about algebraic preliminaries in Chapter 2.

In [Ago08], the author proved that:

Theorem 1.46 ([Ago08])**.** *If M is an irreducible 3-manifold such that its fundamental group $\pi_1(M)$ is infinite and virtually RFRS, then M is virtually fibered.*

At first, since the notion of being RFRS seemed to be very restrictive, it wasn't clear that the Thurston's virtual fibering conjecture could have been proved thanks to this result. But in a few years, thanks to the work of Wise and others, Agol has been able to finally answer to the conjecture [AGM12].

Theorem 1.47 ([AGM12])**.** *The fundamental group of any hyperbolic 3-manifold is virtually RFRS. Hence, any hyperbolic 3-manifold is virtually fibered over the circle.*

However, the proof of this result is not constructive, hence finding a cover which effectively fibers remains an open problem.

In the third chapter we will exhibit a criterion which helps to construct in a more explicit way a tower of finite-degree coverings of arithmetic hyperbolic 3-manifolds (and orbifolds) among which to find the fibered one.

Chapter 2

Algebraic Preliminaries

In the next chapter we will need some background on algebraic number theory. These pages have the aim to define, prove and report results which will come into play later on.

2.1 Number Fields and Algebraic Integers

We start from the very beginning recalling the notion of *number field*.

Definition 2.1. A number field k is a finite extension of $\mathbb{Q}$, i.e. $k = \mathbb{Q}(\{t_i \mid i \in \Omega\})$ where Ω is a finite set and the elements t_i satisfy polynomial conditions with rational coefficients (they are *algebraic* over $\mathbb{Q}$).

Since $\mathbb{Q}$ is a field with characteristic 0, any finite extension is separable and we can apply the Primitive Element Theorem to show that $k = \mathbb{Q}(t)$ is a simple extension. Denoting by $m_t(x)$ the minimum polynomial of t, which is a monic irreducible polynomial with rational coefficients, we have that $m_t(t) = 0$ and $\deg(m_t(x)) = d = [k : \mathbb{Q}]$ the degree of the extension.

If $t = t_1, t_2, \dots, t_d$ are the roots of $m_t(x)$, we can define a field isomorphism $\mathbb{Q}(t) \longrightarrow \mathbb{Q}(t_i)$ by assigning $t \longmapsto t_i$. Viceversa, any field embedding $\sigma\colon k \longrightarrow \mathbb{C}$ maps t to $\sigma(t)$, which again satisfies the same polynomial $m_t(x)$ as t. Thus $\sigma(t)$ is another root of the minimum polynomial of t and there are exactly d field (or *Galois*) embeddings $\sigma_1, \dots, \sigma_d\colon k \longrightarrow \mathbb{C}$. Just like the roots $t_1, \dots, t_d$, which can be real or occur in complex conjugate pairs, the corresponding Galois embeddings can be *real* if $\sigma_i(k) \subseteq \mathbb{R}$ or they appear as couples of conjugate pairs $(\sigma_i, \overline{\sigma_i})$. If the number field k admits r_1 real embeddings and r_2 complex conjugates pairs of Galois embeddings then $d = r_1 + 2r_2$ and we say that k has r_1 *real places* and r_2 *complex places*. Additionally, if $r_2 = 0$ we say that k is *totally real*.

Definition 2.2. Elements in a number field k which satisfy a monic polynomial with coefficients in $\mathbb{Z}$ are called algebraic integers. The set of all algebraic integers in k is denoted by $\mathcal{O}_k$.

Since sums and products of algebraic integers are again algebraic integers, the set $\mathcal{O}_k$ is a commutative ring with unity.

The number field k can be recovered from $\mathcal{O}_k$ as its field of fractions. For if $\alpha \in k$ with minimum polynomial $f(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$ and D is the least common multiple of the denominators of the coefficients, then $D\alpha$ is an algebraic integer because $0 = D^n f(\alpha) = \widetilde{f}(D\alpha)$ with $\widetilde{f}(x) = x^n + Dc_{n-1}x^{n-1} + \ldots + D^{n-1}c_1x + D^n c_0$.

Example 2.3. As an example, we consider the quadratic number field $k = \mathbb{Q}(\sqrt{d})$ with d a square-free integer (not necessarily positive). Then its ring of integers $\mathcal{O}_k$, often denoted by $\mathcal{O}_{-d}$ in this very special case (the reason of the minus sign is due to the fact that in the applications of hyperbolic geometry d is always a negative number), contains $\mathbb{Z}$ and also the element $\sqrt{d}$, since it satisfy the monic polynomial $x^2 - d$. We could ask ourselves whether there are algebraic integers in k other than the ones of the form $a + b\sqrt{d}$ with a, b integers, and the answer is affirmative in the case $d \equiv 1 \pmod 4$: the element $\frac{1+\sqrt{d}}{2}$ satisfies the polynomial $x^2 - x + \frac{1-d}{4}$, which has integral coefficients. We can then prove that $\mathcal{O}_{-d} = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ for $d \equiv 1 \pmod 4$, and $\mathcal{O}_{-d} = \mathbb{Z}\left[\sqrt{d}\right]$ otherwise.

All these rings coming from number fields have some common properties, which now are going to be investigated.

In general factorization of elements in these rings is not unique; for example in $\mathcal{O}_5$ the element 6 can be expressed both as $2 \cdot 3$ and $(1 + \sqrt{-5})(1 - \sqrt{-5})$, and each one of the factors is an irreducible element. What it is true in rings of integers is that ideals have a unique factorization into products of prime ideals. Rings with this property are called *Dedekind Domains*:

Definition 2.4. An integral domain R that is not a field is a *Dedekind domain* if each non-zero proper ideal factors into a product of prime ideals in an unique way.

Equivalently, R is a *Dedekind domain* if it is Noetherian, integrally closed and with Krull dimension 1 (i.e. every non-zero prime ideal is maximal).

Hence, using the second definition, as a result we have that the ring of integers $\mathcal{O}_k$ in a number field k is finitely generated and every quotient $\mathcal{O}_k/\mathfrak{p}$ for $\mathfrak{p}$ prime is a finite field. Thus we can define:

Definition 2.5. Let $\mathfrak{p}$ a prime ideal in $\mathcal{O}_k$, the ring of integers of a number field k. The *norm* $N(\mathfrak{p})$ of the ideal $\mathfrak{p}$ is the number of elements in the finite field $\mathcal{O}_k/\mathfrak{p}$, namely

$$N(\mathfrak{p}) := |\mathcal{O}_k/\mathfrak{p}| .$$

More generally, we can define the norm of any non-zero ideal I of $\mathcal{O}_k$ in the same way as for prime ideals, and it is well defined (i.e. it is finite) due to the fact that $\mathcal{O}_k$ is a Dedekind domain.

2.2 Valuations and Completions

On any field K we can define a valuation, which enriches the space with a metric making it a metric space that could be complete or not. In the latter case we can construct the completion of the metric space as usual by using Cauchy sequences and it can be interesting and useful for our future purposes to work with a complete field instead of a non-complete one.

The results of this section can be found in many introductory texts, such as [MR03], [Bac64] or [Gou97].

So we recall that

Definition 2.6. A *valuation* v on a field K is a mapping $v\colon K \longrightarrow \mathbb{R}_{\geq 0}$ such that:

1. $v^{-1}(0) = \{0\}$
2. v is multiplicative, i.e. $v(xy) = v(x)v(y)$ for all x, y in K;
3. v is subadditive, i.e. $v(x+y) \leq v(x) + v(y)$ for all x, y in K.

Given any valuation v, as said before we can define a metric on K, namely a distance function d defined via $d(x, y) = v(x - y)$. Note that there is always the trivial valuation $v(x) = 1$ for all $x \neq 0$ in K, and this valuation induces the discrete distance.

We will only be interested in a certain type of valuations, *non-Archimedean* ones, because in the setting of number fields they produce completions which result to be the p-adic fields $\mathbb{Q}_p$, in contrast to the complete fields $\mathbb{R}$ and $\mathbb{C}$.

Definition 2.7. A valuation v on a field K is called *non-Archimedean* if, for all x, y in K, it holds that

$$v(x+y) \leq \max\{v(x), v(y)\}.$$

When working with a non-Archimedean valuation v, another ring can be defined. This will be very important in the following study of the completions of number fields with respect to v, altogether with the notion of a *uniformizing element.* In this regard, it holds that:

Lemma 2.8. *Let v be a non-Archimedean valuation on a field K. Let*

$$R(v) := \{\alpha \in K \mid v(\alpha) \leq 1\}$$

and

$$\mathcal{P}(v) := \{\alpha \in K \mid v(\alpha) < 1\}.$$

Then $R(v)$ is a local ring called the valuation ring of K (with respect to v) whose unique maximal ideal is $\mathcal{P}(v)$ and whose field of fractions is K. Moreover, $\mathcal{P}(v)$ is principal.

Note that the valuation ring $R(v)$ is a principal ideal domain, with only ideals of the form $\mathcal{P}(v)^m$ for $m \geq 1$.

We now return to the more specific treatment of number fields: let $K = k$ be a number field and let $\mathcal{O}_k$ its ring of integers, as in the previous section. Then examples of non-Archimedean valuations are the following:

Definition 2.9. Let $\mathfrak{p}$ be any prime ideal in $\mathcal{O}_k$ and let $N(\mathfrak{p})$ its norm. Given any element $x \neq 0$ in $\mathcal{O}_k$ we define

$$v_{\mathfrak{p}}(x) := \left(\frac{1}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(x)}$$

where $n_{\mathfrak{p}}(x)$ is the largest integer m such that $x \in \mathfrak{p}^m$. Since k is the field of fractions of $\mathcal{O}_k$, we can extend the definition of $v_{\mathfrak{p}}$ to k^* by

$$v_{\mathfrak{p}}\left(\frac{x}{y}\right) := \frac{v_{\mathfrak{p}}(x)}{v_{\mathfrak{p}}(y)} = \left(\frac{1}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(x)-n_{\mathfrak{p}}(y)}.$$

It is immediate to prove that $v_{\mathfrak{p}}$ is a valuation, and it is called the $\mathfrak{p}$-*adic valuation*.

We stress the quite trivial fact that the ring of integers $\mathcal{O}_k$ is contained in the valuation ring $R(v_{\mathfrak{p}})$, since $n_{\mathfrak{p}}(x)$ cannot be negative if $x \in \mathcal{O}_k$.

Definition 2.10. Given a number field k with a $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}$, a *uniformizer* or *uniformizing element* is an element $\pi \in \mathcal{O}_k$ such that $n_{\mathfrak{p}}(\pi) = 1$.

From the definition it easily follows that $\mathcal{P}(v_{\mathfrak{p}}) = \pi R(v_{\mathfrak{p}})$. Furthermore, since k is the field of fractions of $\mathcal{O}_k$, the local ring $R(v_{\mathfrak{p}})$ can be identified with the localization of $\mathcal{O}_k$ at the multiplicative set $\mathcal{O}_k \setminus \mathfrak{p}$ and hence k is also the field of fractions of $R(v_{\mathfrak{p}})$. Finally, the *residue field* $R(v_{\mathfrak{p}})/\mathcal{P}(v_{\mathfrak{p}}) = R(v_{\mathfrak{p}})/\pi R(v_{\mathfrak{p}})$ coincides with $\mathcal{O}_k/\mathfrak{p}$.

A first natural question about number fields endowed with $\mathfrak{p}$-adic metrics is whether they are complete or not. Recall the definition of complete metric space:

Definition 2.11. A metric space is *complete* if every Cauchy sequence converges to an element of the space.

It can be proved that:

Proposition 2.12. *A number field k endowed with a valuation v is never complete. However, there exists a unique up to valuation-preserving isomorphisms field k_v in which k embeds, such that the valuation v extends to k_v and k_v is complete with respect to this extended valuation. The field k_v is called the completion of k at v.*

We observe that since the valuation v on k extends to a valuation $\widetilde{v}$ on k_v, we have an inclusion of the valuation ring $R(v)$ inside the valuation ring $R(\widetilde{v})$ of k_v with respect to the extended valuation $\widetilde{v}$.

If $v = v_\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_k$, the completion is usually denoted as $k_\mathfrak{p}$ and it is referred to as a $\mathfrak{p}$-*adic field*. As in the non-complete setting, the valuation ring of $k_\mathfrak{p}$ is a local ring whose unique prime ideal is principal. More precisely, the following holds:

Proposition 2.13. *Let $R_\mathfrak{p}$ denote the valuation ring of the complete field $k_\mathfrak{p}$ and let $i_\mathfrak{p}$ denote an embedding of k in $k_\mathfrak{p}$. Then $R_\mathfrak{p}$ is a local ring and its unique maximal ideal is generated by $i_\mathfrak{p}(\pi)$, where $\pi \in \mathcal{O}_k$ (or equivalently in $R(v_\mathfrak{p})$) is an uniformizing element. Furthermore, the three residue fields $R_\mathfrak{p}/i_\mathfrak{p}(\pi)R_\mathfrak{p}$, $R(v_\mathfrak{p})/\pi R(v_\mathfrak{p})$ and $\mathcal{O}_k/\mathfrak{p}$ all coincide.*

The uniformizing element in the completion can be used to give an explicit description of the elements of the $\mathfrak{p}$-adic field $k_\mathfrak{p}$ as power series. Let $\{c_i\}$ be a set of coset representatives of the ideal $\pi R_\mathfrak{p}$ in $R_\mathfrak{p}$, where actually instead of π there should be $i_\mathfrak{p}(\pi)$, but for the sake of simplicity we will omit the embedding. These coset representatives can be identified with the elements of the residue field, thus the cardinality of the set $\{c_i\}$ is $N(\mathfrak{p})$. Moreover, the set is chosen so that the zero coset is represented by 0. Then:

Theorem 2.14. *Every element $\alpha \neq 0$ in $k_\mathfrak{p}$ has a unique expression in the form*

$$\alpha = \pi^r \sum_{n=0}^{\infty} c_{i_n} \pi^n$$

where $r \in \mathbb{Z}$ and $c_{i_0} \neq 0$.

Furthermore, we can say something more precise about what the completions really are, and to do this we rely on the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to its p-adic valuation, which will be studied in detail in the next section.

Theorem 2.15. *The completion $k_\mathfrak{p}$ of a number field k at the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$ is isomorphic to an extension of the field of p-adic numbers $\mathbb{Q}_p$, where p is the characteristic of the residue field $\mathcal{O}_k/\mathfrak{p}$. In particular, the degree of the extension is $n = ef$ where e is the ramification index of $\mathfrak{p}$ over p and f is its inertial degree. Explicitly, e is the exponent of $\mathfrak{p}$ in the factorization into prime ideals of the ideal $p\mathcal{O}_k$, while p^f is the cardinality of the residue field $\mathcal{O}_k/\mathfrak{p}$, or equivalently $p^f = N(\mathfrak{p})$.*

Without getting lost in definitions and results that would lead us off-track, we give the next result, due to Kummer, which characterizes in a more practical way the factorization of $p\mathcal{O}_k$ into prime ideals and also the inertial degrees:

Theorem 2.16 (Kummer)**.** *Let k be a number field and $\mathcal{O}_k = \mathbb{Z}[\theta]$ for some $\theta \in \mathcal{O}_k$. Suppose that $m_\theta(x)$ is the minimum polynomial of θ and p is a prime number. By reducing the coefficients modulo $\mathbb{F}_p$ we obtain a factorization*

$$\overline{m_\theta}(x) = \overline{h_1}(x)^{e_1} \cdot \ldots \overline{h_r}(x)^{e_r}$$

where $h_i(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree f_i. Then $\mathfrak{p}_i = (p, h_i(\theta))$ is a prime ideal, its norm is $N(\mathfrak{p}_i) = p^{f_i}$ and

$$p\mathcal{O}_k = \mathfrak{p}_1^{e_1} \ldots \mathfrak{p}_r^{e_r}.$$

The previous theorem has an easy application when we are talking about quadratic extensions of $\mathbb{Q}$:

Corollary 2.17. *Let $k = \mathbb{Q}(\sqrt{d})$ with d a square-free integer be a quadratic extension of $\mathbb{Q}$ and let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_k$. If p is the characteristic of the finite field $\mathcal{O}_k/\mathfrak{p}$, then $k_\mathfrak{p} \cong \mathbb{Q}_p$ if and only if one of the following holds:*

1. *$p = 2$ and $d \equiv 1 \pmod 8$, or*

2. *p is odd and d is a non-zero quadratic residue modulo p.*

Proof. Recall from Example 2.3 that the ring of algebraic integers in a quadratic number field depends on the remainder of the division of d by 4. Suppose first that $d \not\equiv 1 \pmod 4$; we will apply Theorem 2.16 with $\theta = \sqrt{d}$ and $m_\theta(x) = x^2 - d$. We analyze case by case every possibility.

If the prime number p is 2, then by reducing the minimal polynomial modulo 2 we obtain the factorization $\overline{m_\theta}(x) = (x - \overline{d})^2$, from which we have that $\mathfrak{p} = (2, \sqrt{d} - d)$ and that it has ramification index $e = 2$ and inertial degree $f = 1$. In the same way, if p is odd and $p|d$ we obtain $\overline{m_\theta}(x) = x^2$, $\mathfrak{p} = (2, \sqrt{d})$, $e = 2$ and $f = 1$. In both cases the completion $k_\mathfrak{p}$ is a quadratic extension of $\mathbb{Q}_p$.

If $p \nmid 2d$, the minimal polynomial $\overline{m_\theta}(x)$ factorizes into a product of two distinct degree-1 factors if and only if d is a quadratic residue modulo p. If it factorizes then $\mathfrak{p}$ can be $(p, \sqrt{d} - d)$ or $(p, \sqrt{d} + d)$ and for both $e = f = 1$. If it does not factor then $\mathfrak{p} = (p)$ and its inertial degree is $f = 2$.

Suppose now that $d \equiv 1 \pmod 4$; we have $\theta = \frac{1+\sqrt{d}}{2}$ and $m_\theta(x) = x^2 - x + \frac{1-d}{4}$.

If the prime number p is 2, then we have two possibilities depending on the remainder of d modulo 8: if $d \equiv 1 \pmod 8$ the reduced minimal polynomial factorizes as $\overline{m_\theta}(x) = x^2 + x = x(x+1)$, $\mathfrak{p}$ can be $(2, \theta)$ or $(2, \theta + 1)$ and for both $e = f = 1$; if instead $d \equiv 5 \pmod 8$, the polynomial $\overline{m_\theta}(x) = x^2 + x + 1$ is irreducible, hence $\mathfrak{p} = (2, \theta^2 + \theta + 1)$ and its inertial degree is $f = 2$.

If p is an odd prime number, $\overline{m_\theta}(x)$ has the same roots as $(2x - 1)^2 - \overline{d}$. If $p|d$, as before we obtain that the prime ideal $\mathfrak{p}$ has ramification index $e = 2$. Again, if $p \nmid 2d$ the minimal polynomial $\overline{m_\theta}(x)$ factorizes if and only if d is a quadratic residue modulo p. If it factorizes then $\mathfrak{p}$ has $e = f = 1$, otherwise $\mathfrak{p}$ has inertial degree $f = 2$.

Summing up, we can schematize what we proved in the following:

1. Let p be odd:

    - If $p|d$, the prime ideal $\mathfrak{p}$ has ramification index $e = 2$;
    - If d is a non-zero quadratic residue modulo p, the prime ideal $\mathfrak{p}$ has $e = f = 1$;
    - If d is a non-zero quadratic non-residue modulo p, the prime ideal $\mathfrak{p}$ has inertial degree $f = 2$.

2. Let $p = 2$:

    - If $d \not\equiv 1 \pmod 4$, the prime ideal $\mathfrak{p}$ has ramification index $e = 2$;

- If $d \equiv 1 \pmod 8$, the prime ideal $\mathfrak{p}$ has $e = f = 1$;
- If $d \equiv 5 \pmod 8$, the prime ideal $\mathfrak{p}$ has inertial degree $f = 2$.

Hence the completion $k_\mathfrak{p}$ is a quadratic extension of $\mathbb{Q}_p$ in any case different from those stated in the corollary, and coincides with $\mathbb{Q}_p$ in those cases. □

When the completion $k_\mathfrak{p}$ of a number field k with respect to the $\mathfrak{p}$-adic valuation is isomorphic to $\mathbb{Q}_p$, it is also easy to find the embeddings $i_\mathfrak{p}\colon k \longrightarrow k_\mathfrak{p} \cong \mathbb{Q}_p$. Through the isomorphisms $k \cong \mathbb{Q}(t) \cong \mathbb{Q}[x]/\left(m_t(x)\right)$ given by the Primitive Element Theorem, embeddings $i_\mathfrak{p}$ are induced by the choice of a root of the polynomial $m_t(x)$ inside $\mathbb{Q}_p$.

We will show shortly how to find roots in the field of p-adic numbers.

2.3 The field of p-adic numbers $\mathbb{Q}_p$

In this section we are going to present the field of p-adic numbers $\mathbb{Q}_p$ and show some properties which can help to handle the elements of this field. There are many introductory books about this topic, see for example [Bac64] or [Gou97]. The classical rigorous definition of $\mathbb{Q}_p$ is the following:

Definition 2.18. Given a prime number p, the field of *p-adic numbers* $\mathbb{Q}_p$ is defined to be the completion of $\mathbb{Q}$ with respect to the p-adic valuation, i.e. the set of Cauchy sequences in $\mathbb{Q}$ modulo the ideal of null sequences $\mathcal{N} = \{(x_n) \mid \lim_n v_p(x_n) = 0\}$.

Obviously the abstract definition is not what we were searching for to understand what the elements actually are. Thus, taking the (trivial) number field $\mathbb{Q}$, with ring of integers $\mathcal{O}_\mathbb{Q} = \mathbb{Z}$ and prime ideal (p), we can choose as an uniformizing element $\pi = p$ and apply Theorem 2.14:

Corollary 2.19. *Every element $\alpha \neq 0$ in $\mathbb{Q}_p$ can be written as*

$$\alpha = p^r \sum_{n=0}^{\infty} a_n p^n$$

in a unique way, where $r \in \mathbb{Z}$, each a_i is in $\{0, 1, \ldots, p-1\}$ and $a_0 \neq 0$.

We will refer to this expression as the p-adic expansion of α.

With this characterization it is a little more sophisticated to find the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ than with the definition, in which $\mathbb{Q}$ is easily identified with the set of constant Cauchy sequences. We will see now how this embedding can be seen to understand better the p-adic expansions.

Everything relies only on the treatment of formal power series in one variable, especially when talking about inverses. Let's start from natural numbers: each of them can be written in a unique way as a finite sum

$$n = a_0 + a_1 p + \ldots + a_m p^m$$

where $0 \leq a_i \leq p-1$ for all i.

Now, we can also extend the p-adic expansion to positive rational numbers: using the p-adic expression for both the numerator and the denominator we can apply the formal rules for polynomial inverses to obtain a power series representing the positive rational number.

Example 2.20. Let us fix $p = 3$. Consider for example the positive rational number $\frac{19}{23}$. Then we have

$$19 = 1 + 0 \times 3 + 2 \times 3^2 = 1 + 2p^2$$

and

$$23 = 2 + 1 \times 3 + 2 \times 3^2 = 2 + p + 2p^2.$$

Then the 3-adic expansion of $\frac{19}{23}$ is

$$\frac{19}{23} = \frac{1+2p^2}{2+p+2p^2} = 2 + p^3 + 2p^4 + 2p^7 + p^8 + p^9 + \ldots$$

It is easy to verify the above equality by computing the product of that power series with $2 + p + 2p^2$, remembering that for every k we have $p^k + p^k + p^k = p^{k+1}$.

It remains to see what is the p-adic expansion of a negative rational number. Since power series can be multiplied and the multiplication operation is compatible with the one defined on $\mathbb{Q}$, it is sufficient to find the p-adic expression of -1.

The natural writing of this element is

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \ldots = \sum_{i=0}^{\infty}(p-1)p^i,$$

because each term on the right hand side but -1 cancels out in the power series.

Thus, the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ can be seen by mapping each rational x to its p-adic expansion according to the constructions made above.

But another characterization holds, at least for elements in the valuation ring of $\mathbb{Q}_p$, and it can help a lot when searching for roots of integer polynomials in $\mathbb{Q}_p$. Indeed, note that having written elements in their p-adic expansion allows us to easily compute their p-adic norm:

Lemma 2.21. *If $\alpha \neq 0$ in $\mathbb{Q}_p$ and following the same notations of Corollary 2.19*

$$\alpha = p^r \sum_{n=0}^{\infty} a_n p^n,$$

then $v_p(\alpha) = p^{-r}$. This implies that

$$R_p = \{\alpha \in \mathbb{Q}_p \mid v_p(\alpha) \leq 1\} = \left\{ \alpha = p^r \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p \;\middle|\; r \geq 0 \right\}.$$

Hence every element $\alpha = \sum_{i=0}^{\infty} a_i p^i$ in the valuation ring R_p (here we relax the hypotesis on a_0, allowing it to be 0) can be approximated by a Cauchy sequence contained in $\mathbb{Z}$. If we consider the sequence of partial sums $(x_n)_{n\geq 1}$, where

$$x_n = \sum_{i=0}^{n-1} a_i p^i,$$

then

$$d(\alpha, x_n) = v_p \left(p^n \sum_{i=0}^{\infty} a_{n+i} p^i \right) \leq p^{-n}.$$

These partial sums suggest the characterization we were searching for, since they follow the relation

$$x_n \equiv x_m \pmod{p^m}$$

for $n \geq m$. We finally come to:

Theorem 2.22. *Let $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^k\mathbb{Z}$ be the ring of coherent sequences, that is to say sequences $(x_n)_{n\geq 1}$ of elements $x_n \in \mathbb{Z}/p^n\mathbb{Z}$ for which $x_n \equiv x_m \pmod{p^m}$ each time $n \geq m$. Then we have*

$$R_p \cong \mathbb{Z}_p$$

with isomorphism given by taking the sequence of partial sums in the p-adic expansion of elements in R_p.

We close this section giving an algorithmic method to find roots of integer polynomials in $\mathbb{Q}_p$. Using the last characterization of elements in the valuation ring (which is exactly the completion of $\mathbb{Z}$ with respect to the p-adic valuation), given a polynomial $F(x) \in \mathbb{Z}[x]$ we can rephrase the problem of finding roots in $\mathbb{Q}_p$ in terms of coherent sequences.

Indeed, a root of the polynomial $F(x)$ can be found (if it exists) by computing recursively a sequence $(x_n)_{n\geq 1}$ such that $0 \leq x_n \leq p^n - 1$ and

$$\begin{cases} F(x_n) \equiv 0 \pmod{p^n} \\ x_n \equiv x_{n-1} \pmod{p^{n-1}}. \end{cases}$$

Let's see an example of application of this method, which will return into play later on in the next chapter.

Example 2.23. Let $k = \mathbb{Q}(\sqrt{-7})$ and $F(x) = x^2 - x + 2$. The polynomial $F(x)$ has no roots in $\mathbb{Q}$, while its roots in $\mathbb{C}$ are $\frac{1\pm\sqrt{-7}}{2}$. Moreover, it is well known and it has been already said in Example 2.3 that the ring of integers $\mathcal{O}_7$ is $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. Hence, if we consider $\mathfrak{p} = \left(\frac{1+\sqrt{-7}}{2}\right)$ and $p = 2$, we can apply Corollary 2.17 which implies, since $-7 \equiv 1 \pmod 8$,

$$k_{\mathfrak{p}} \cong \mathbb{Q}_2.$$

For a better understanding, we can also retrace the route from Theorem 2.15 and Theorem 2.16 directly with our example: defining $\theta = \frac{1+\sqrt{-7}}{2}$, its minimal polynomial $m_\theta(x)$ coincides with $F(x)$ and its reduction modulo 2 is

$$\overline{F}(x) \equiv x^2 - x \equiv x^2 + x \equiv x(x+1) =: \overline{h_1}(x)\overline{h_2}(x) \pmod 2.$$

The prime $\mathfrak{p} = \left(\frac{1+\sqrt{-7}}{2}\right) = (2, h_1(\theta))$ has ramification index $e = 1$ and inertial degree $f = 1$, therefore the completion $k_\mathfrak{p}$ is an extension of $\mathbb{Q}_2$ of degree $n = ef = 1$, i.e. $k_\mathfrak{p} = \mathbb{Q}_2$.

As explained above, to find roots of $F(x)$ in $\mathbb{Q}_2$ we need to solve some modular congruences lifting the solutions recursively.

Thus, solving the equation $F(x) \equiv 0 \pmod 2$ we find that both $x \equiv 0 \pmod 2$ and $x \equiv 1 \pmod 2$ are allowable solutions. Lifting these solutions modulo 4 and then doing the same recursively modulo 2^n we have

$$\begin{array}{lll}
x \equiv 0,1,2,3 \pmod{2^2}: & F(0) \not\equiv 0 \pmod{2^2} & F(2) \equiv 0 \pmod{2^2} \\
 & F(1) \not\equiv 0 \pmod{2^2} & F(3) \equiv 0 \pmod{2^2} \\
x \equiv 2,3,6,7 \pmod{2^3}: & F(2) \not\equiv 0 \pmod{2^3} & F(6) \equiv 0 \pmod{2^3} \\
 & F(3) \equiv 0 \pmod{2^3} & F(7) \not\equiv 0 \pmod{2^3} \\
x \equiv 3,6,11,14 \pmod{2^4}: & F(3) \not\equiv 0 \pmod{2^4} & F(11) \equiv 0 \pmod{2^4} \\
 & F(6) \equiv 0 \pmod{2^4} & F(14) \not\equiv 0 \pmod{2^4} \\
x \equiv 6,11,22,27 \pmod{2^5}: & F(6) \equiv 0 \pmod{2^5} & F(22) \not\equiv 0 \pmod{2^5} \\
 & F(11) \not\equiv 0 \pmod{2^5} & F(27) \equiv 0 \pmod{2^5} \\
x \equiv 6,27,38,59 \pmod{2^6}: & F(6) \not\equiv 0 \pmod{2^6} & F(38) \equiv 0 \pmod{2^6} \\
 & F(27) \equiv 0 \pmod{2^6} & F(59) \not\equiv 0 \pmod{2^6} \\
x \equiv 27,38,91,102 \pmod{2^7}: & F(27) \not\equiv 0 \pmod{2^7} & F(91) \equiv 0 \pmod{2^7} \\
 & F(38) \equiv 0 \pmod{2^7} & F(102) \not\equiv 0 \pmod{2^7}
\end{array}$$

and so on. At each step we lifted 2 solutions modulo 2^n to 4 possible solutions modulo 2^{n+1}, but only 2 of them were actually solutions modulo 2^{n+1}. We ask ourselves if this heuristic property that we checked for $n \leq 6$ remains true for every bigger value of n. If this claim holds, repeating this procedure we can construct exactly two solutions of $F(x)$ in $\mathbb{Q}_2$, which are

$$x_1 = 2 + 2^2 + 2^5 + \dots, \qquad x_2 = 1 + 2 + 2^3 + 2^4 + 2^6 + \dots$$

Notice that $x_2 \in \{\alpha \in \mathbb{Q}_2 \mid v_2(\alpha) = 1\} = \mathbb{Z}_2^*$, i.e. x_2 is invertible in the ring of 2-adic integers, while $v_2(x_1) = 2^{-1}$.

It remains to prove the claim, which is an application of the following modular arithmetic result:

Proposition 2.24 (Hensel's Lemma [Gou97])**.** *Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integral coefficients and let $f'(x)$ be its formal derivative. Suppose that $a \in \mathbb{Z}$ satisfies $f(a) \equiv 0 \pmod{p^j}$ and $f'(a) \not\equiv 0 \pmod{p}$. Then there exists a unique t modulo p such that $f(a + tp^j) \equiv 0 \pmod{p^{j+1}}$.*

As a consequence of this, by applying the statement recursively we have that if an integer a is such that $f(a) \equiv 0 \pmod{p}$ and $f'(a) \not\equiv 0 \pmod{p}$, then there exists a unique p-adic integer $\alpha \in \mathbb{Z}_p$ such that $f(\alpha) = 0$ and $\alpha \equiv a \pmod{p}$.

In our example, since $F'(x) = 2x - 1$ has no roots modulo 2, we can apply Hensel's Lemma to lift the two modulo 2 roots $a \equiv 0, 1 \pmod{2}$ to the solutions $x_1, x_2 \in \mathbb{Z}_2$ and that these are the only solutions of $F(x)$ in $\mathbb{Z}_2$.

2.4 Algebraic Groups and Arithmetic Hyperbolic Manifolds

In this section we will present some definitions arising naturally from algebraic geometry and Lie theory which help us to introduce the notion of arithmetic Kleinian group and hence of arithmetic hyperbolic 3-manifold. These manifolds form a rich and very interesting family and the "arithmeticity" property will play a central role in the construction cosidered in the next chapter.

We start with the definition of *algebraic group*:

Definition 2.25. An *algebraic group* is an algebraic variety with the multiplication and inversion operations which are *regular maps*. In other words, an algebraic group is a group defined by polynomial relations in an affine space and the two operations are given locally by rational functions.

A *linear algebraic group* $\mathcal{G}$ is a subgroup of a group of determinant-1 complex matrices defined by polynomial equations. If the polynomials have coefficients in a field K, $\mathcal{G}$ is said to be defined over K.

Observe that we can identify $\mathrm{GL}(n, \mathbb{C})$ with a linear algebraic group inside $\mathrm{SL}(n + 1, \mathbb{C})$: indeed it is easy to see that the subgroup of matrices $M = (M_{ij}) \in \mathrm{SL}(n+1, \mathbb{C})$ with entries $M_{1j} = M_{i1} = 0$ for $2 \leq i, j \leq n + 1$ is isomorphic to $\mathrm{GL}(n, \mathbb{C})$. Hence any subgroup of $\mathrm{GL}(n, \mathbb{C})$ defined by polynomial equations is a linear algebraic group.

We have already seen that a model of the hyperbolic space $\mathbb{H}^3$ is the half-space H^3, and in that case the orientation-preserving isometries act on the boundary sphere as Möbius transformations and form a group with the composition operation which is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$.

This group, which is clearly the same as $\mathrm{PGL}(2, \mathbb{C})$, turns out to be a linear algebraic group, even though not in a trivial way. Indeed:

Proposition 2.26. *The projective linear group* $\mathrm{PGL}(2,\mathbb{C})$ *is isomorphic to the group of* $\mathbb{C}$*-algebra isomorphisms* $\mathrm{Aut}(\mathrm{M}(2,2,\mathbb{C}))$*, which is a linear algebraic group contained in* $\mathrm{GL}(4,\mathbb{C})$*.*

Proof. Let us define the following homomorphism:

$$\begin{array}{cccc} \varphi\colon & \mathrm{GL}(2,\mathbb{C}) & \longrightarrow & \mathrm{Aut}(\mathrm{M}(2,2,\mathbb{C})) \\ & A & \longmapsto & (M \mapsto AMA^{-1}) \end{array}$$

It is easy to check that $\mathrm{Ker}\,\varphi$ consists of the center of $\mathrm{GL}(2,\mathbb{C})$, i.e. scalar multiples of the identity matrix. The surjectivity of φ is due to the Skolem-Noether Theorem [MR03], which ensures that every automorphism is an inner one.

Now, in order to see that $\mathrm{Aut}(\mathrm{M}(2,2,\mathbb{C}))$ is a linear algebraic group inside $\mathrm{GL}(4,\mathbb{C})$, it is sufficient to establish a $\mathbb{C}$-basis of $\mathrm{M}(2,2,\mathbb{C})$ and to write the automorphisms (which are linear isomorphisms) with respect to this basis. Then it holds that $\mathrm{Aut}(\mathrm{M}(2,2,\mathbb{C}))$ is defined by polynomial equations, since it consists of all those linear isomorphisms F such that for all matrices $M, N \in \mathrm{M}(2,2,\mathbb{C})$ we have $F(M)F(N) = F(MN)$, and this condition is polynomial in the entries of F. $\square$

We also recall that orientable hyperbolic 3-manifolds are quotients $\mathbb{H}^3/\Gamma$ with $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ a torsion-free Kleinian group. In this thesis, as mentioned earlier, we will focus on the family of *arithmetic hyperbolic 3-manifolds*, whose definition relies on properties of the fundamental group Γ linked to what we studied earlier about number fields in this chapter. Arithmetic manifolds are quotients of $\mathbb{H}^3$ by *arithmetic subgroups* of $\mathrm{PSL}(2,\mathbb{C})$.

We will need the following notation:

Definition 2.27. A linear algebraic group is said to be *semisimple* if it is connected and it doesn't contain non-trivial abelian, connected, closed, normal subgroups.

We are ready to present arithmetic groups. The definition comes from the idea of taking matrices with integral entries inside a group, and then generalizing it by allowing the natural relation of commensurability between subgroups and also by allowing different representations of a group inside $\mathrm{GL}(n,\mathbb{C})$. We then obtain the following definition:

Definition 2.28. Let $\mathcal{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$ and let Γ be a subgroup of the set of rational points $\mathcal{G}_{\mathbb{Q}}$ of $\mathcal{G}$, that is to say elements with rational coefficients. Then Γ is an *arithmetic subgroup* of $\mathcal{G}$ if for a $\mathbb{Q}$-representation $\rho\colon \mathcal{G} \longrightarrow \mathrm{GL}(n,\mathbb{C})$, i.e. a group homomorphism which is also a morphism between algebraic groups defined over $\mathbb{Q}$, the image $\rho(\Gamma)$ is commensurable with the set of integer points $\rho(\mathcal{G})_{\mathbb{Z}}$ of the image $\rho(\mathcal{G})$, namely the set of matrices in $\rho(\mathcal{G})$ with entries in $\mathbb{Z}$.

More generally, we could extend the definition above by considering a group $\mathcal{G}$ defined over a number field k and a subgroup Γ of $\mathcal{G}_k$ such that $\rho(\Gamma)$ is commensurable with $\rho(\mathcal{G})_{\mathcal{O}_k}$. It turns out that this wouldn't be a true generalization, because of an argument called *restriction of scalars*. Indeed, it is possible to construct a linear algebraic group

$\mathcal{H}$ defined over $\mathbb{Q}$ such that we have isomorphisms $\mathcal{H}_{\mathbb{Q}} \cong \mathcal{G}_k$ and $\mathcal{H}_{\mathbb{Z}} \cong \mathcal{G}_{\mathcal{O}_k}$.

In the specific setting of Kleinian groups, we could say something more precise about arithmetic Kleinian groups. In [MR03] we can find a specific investigation about invariant trace fields, invariant quaternion algebras and an alternative construction of these groups, although we won't need it in the rest of this thesis.

At this point of the work, we can finally define arithmetic hyperbolic manifolds:

Definition 2.29. An *arithmetic hyperbolic manifold* is the quotient $\mathbb{H}^3/\Gamma$ where Γ is an arithmetic Kleinian group.

Something which we will need is the notion of *congruence subgroup*:

Definition 2.30. Let $\mathcal{G}$ be a linear algebraic group defined over a number field k. If I is an ideal of $\mathcal{O}_k$, the *principal congruence subgroup of level I* of $\mathcal{G}_{\mathcal{O}_k}$ is

$$\mathcal{G}_{\mathcal{O}_k}(I) := \operatorname{Ker} \pi_I,$$

where π_I is the restriction of the projection $M(n, n, \mathcal{O}_k) \twoheadrightarrow M(n, n, \mathcal{O}_k/I)$ induced by the reduction modulo I of the elements' entries and n is the order of the matrices in $\mathcal{G}$.

Definition 2.31. A *congruence subgroup* of a linear algebraic group $\mathcal{G}$ is a subgroup which contains a principal congruence subgroup.

Observe that, since $\mathcal{O}_k$ is a Dedekind domain, the ring $\mathcal{O}_k/I$ is finite whenever $I \neq 0$. Thus every congruence subgroup is of finite index in $\mathcal{G}_{\mathcal{O}_k}$.

Having in mind from the previous chapter the notion of commensurability, we close this section defining an object that will be crucial in our study of arithmetic hyperbolic 3-manifolds.

Definition 2.32. Let G be a group and fix a subgroup Γ. The *commensurator* of Γ in G, denoted as $\mathrm{Comm}_G(\Gamma)$, is the set of elements g in G such that the conjugate subgroup $g\Gamma g^{-1}$ is commensurable with Γ.

It holds that:

Proposition 2.33. *Let $\mathcal{H} \leq \mathrm{GL}(n, \mathbb{C})$ be a linear algebraic group defined over a number field k. Then, the set of k-points of $\mathcal{H}$, namely $\mathcal{H}_k$, commensurates $\mathcal{H}_{\mathcal{O}_k}$, i.e. for every $\gamma \in \mathcal{H}_k$ the group $\gamma \mathcal{H}_{\mathcal{O}_k} \gamma^{-1}$ is commensurable with $\mathcal{H}_{\mathcal{O}_k}$. In other words we have that*

$$\mathcal{H}_k \subseteq \mathrm{Comm}_{\mathcal{H}}(\mathcal{H}_{\mathcal{O}_k}).$$

Proof. Given any $\gamma \in \mathcal{H}_k$, since k is the field of fractions of $\mathcal{O}_k$ there exists an element $m \in \mathcal{O}_k$ such that both $m\gamma$ and $m\gamma^{-1}$ have entries in $\mathcal{O}_k$ (it is sufficient to take a multiple of each denominator for the coefficients of the two matrices). Then we have that $\gamma\Gamma(m^2)\gamma^{-1} \subseteq \mathcal{H}_{\mathcal{O}_k}$, where

$$\Gamma(m^2) = \left\{ M \in \mathcal{H}_{\mathcal{O}_k} \;\middle|\; M \equiv I \pmod{m^2} \right\}.$$

Indeed, if we write any element in $\Gamma(m^2)$ as $M = I + m^2 N$ with the coefficients of N in $\mathcal{O}_k$, then it holds that

$$\gamma M \gamma^{-1} = \gamma(I + m^2 N)\gamma^{-1} = I + (m\gamma)N(m\gamma^{-1})$$

is a matrix in $\mathcal{H}$ with coefficients in $\mathcal{O}_k$. With the same argument we also have that $\gamma^{-1}\Gamma(m^2)\gamma \subseteq \mathcal{H}_{\mathcal{O}_k}$ and hence:

$$\Gamma(m^2) \leq \mathcal{H}_{\mathcal{O}_k} \cap \gamma\mathcal{H}_{\mathcal{O}_k}\gamma^{-1} \leq \mathcal{H}_{\mathcal{O}_k},$$

$$\gamma\Gamma(m^2)\gamma^{-1} \leq \mathcal{H}_{\mathcal{O}_k} \cap \gamma\mathcal{H}_{\mathcal{O}_k}\gamma^{-1} \leq \gamma\mathcal{H}_{\mathcal{O}_k}\gamma^{-1},$$

proving that the intersection is a congruence subgroup of both and thus is of finite index in both. This completes the proof. □

If we combine this result with the following proposition, it follows that the commensurator $\mathrm{Comm}_{\mathcal{H}}(\mathcal{H}_{\mathcal{O}_k})$ of the arithmetic subgroup $\mathcal{H}_{\mathcal{O}_k}$ is dense in $\mathcal{H}$:

Proposition 2.34 ([Mor15]). *Let $\mathcal{H}$ be a linear algebraic group. Then it is defined over the number field k if and only if $\mathcal{H}_k$ is dense in $\mathcal{H}$.*

A converse of the last observation also holds. It is due to Margulis, and characterizes the arithmetic groups as those with a dense commensurator. The arithmeticity hypothesis of a subgroup $\Gamma \leq G$ will help us in the next chapter, when we will need to find an infinite sequence of elements inside $\mathrm{Comm}_G(\Gamma)$.

Theorem 2.35 ([Mor15, 16.3.3]). *Let $\mathcal{G}$ be a semisimple linear algebraic group with no compact factors and let Γ be an irreducible discrete subgroup of finite covolume, i.e. ΓN is dense in $\mathcal{G}$ for every non-compact, closed, normal subgroup N of $\mathcal{G}$. Then Γ is arithmetic if and only if the commensurator $\mathrm{Comm}_{\mathcal{G}}(\Gamma)$ of Γ is dense in $\mathcal{G}$.*

2.5 Coxeter Complexes and Buildings

To understand the structure of an algebraic group in a geometric manner, Tits [Tit79] introduced in 1950s the notion of *spherical buildings* (or Tits buildings). During the years the original definition was generalized in various ways, and among them the *Euclidean buildings* (or Bruhat-Tits buildings) are the topic of this section. See [AB08] for proofs and a deeper treatment of buildings theory.

Let's start by defining Coxeter groups and Coxeter complexes:

Definition 2.36. A Coxeter group W is a group with a presentation $\langle S \mid R\rangle$ where S is a finite set of generators and $R = \left\{(st)^{m(s,t)}\right\}_{s,t\in S}$ is the set of relations, with $m(s,s) = 1$ for every s and the convention $m(s,t) = \infty$ if there are no relations between s and t. Explicitly, we have the relations s^2 for every $s \in S$ and every other relation in R is of the form $(st)^m$ for some $m \geq 1$. We also refer to the couple (W, S) as a Coxeter system.

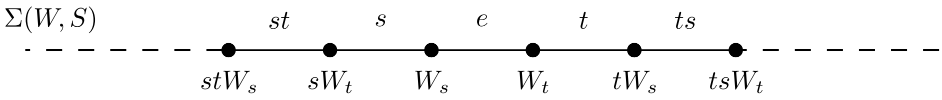

Figure 2.1: The Coxeter complex associated to the infinite dihedral group.

Definition 2.37. Given any subset $T \subseteq S$ let W_T denote the subgroup $\langle T \rangle$. If we consider the poset $P(W,S)$ whose elements are the cosets $\{wW_T \mid w \in W,\ T \subsetneq S\}$ with ordering induced by reverse inclusion, we define the Coxeter complex $\Sigma(W,S)$ associated to the Coxeter system (W,S) as the realization as a simplicial complex of this poset.

With realization we mean that $\Sigma(W,S)$ is a simplicial complex whose simplices are in one-to-one correspondence with elements of $P(W,S)$ and the bijection is a poset isomorphism when considering $\Sigma(W,S)$ itself a poset with partial ordering $\leq_\Sigma$ given by the face relations, i.e. $\tau \leq_\Sigma \sigma$ if τ is a (iterated) face of σ.

It is not true in general that any given poset can be realized as a simplicial complex, but

Proposition 2.38. *Let $(P, \leq_P)$ be a poset. If it satisfies the two properties*

- *Any two elements with a lower bound have a greatest lower bound;*
- *For any element, the poset of its lower bounds is isomorphic to the poset of subsets of $\{1,\dots,r\}$ for some integer $r \geq 0$. In this case we say that the element has rank r;*

then there exists a poset isomoprhism between the poset P and a simplicial complex, called its realization.

Proof. The realization is defined as follows: let the vertex set V be the set of rank-1 elements of the poset P, and then for each $A \in P$ we define the simplex $A' = \{v \in V \mid v \leq_P A\}$. Considering the complex with the poset ordering given by the face relations, it is immediate to check that $A \longmapsto A'$ defines a poset isomorphism and that properties 1. and 2. ensure that this realization is simplicial. □

We briefly visualise the construction of Coxeter complexes with two examples.

Example 2.39. Let $W = \langle s, t \mid s^2, t^2 \rangle$ be the infinite dihedral group. Then its poset of standard cosets is

$$P(W,S) = \{wW_\emptyset,\ wW_s,\ wW_t\ |w \in W\}$$

with order given by the reverse inclusion. In $\Sigma(W,S)$ we have the vertices labeled with the cosets W/W_s and W/W_t, while the 1-simplexes are the cosets $W/W_\emptyset = W$. Attaching everything we obtain the simplicial complex in Figure 2.1, which is the tiling of the Euclidean line by infinite 1-simplices.

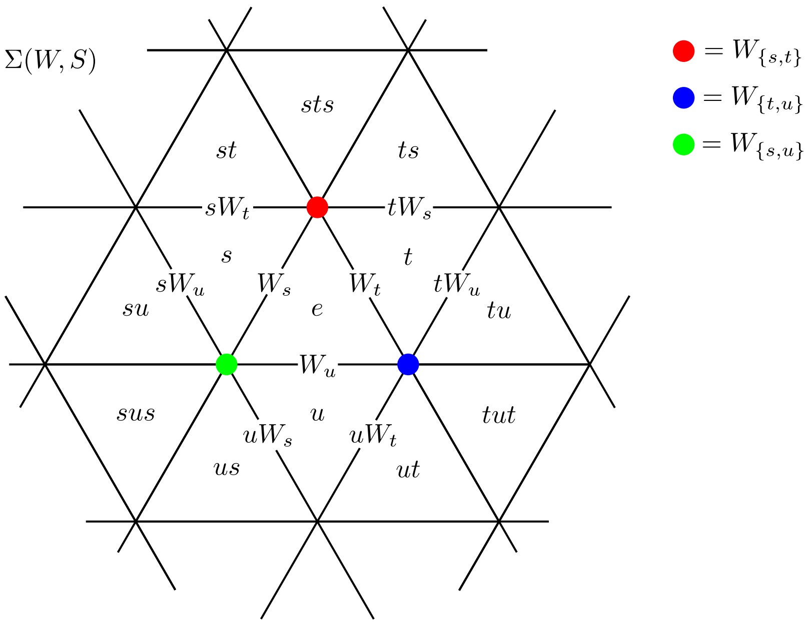

Figure 2.2: The Coxeter complex associated to $W = \langle s, t, u \mid s^2, t^2, u^2, (st)^3, (tu)^3, (su)^3 \rangle$.

Example 2.40. Let $W = \langle s, t, u \mid s^2, t^2, u^2, (st)^3, (tu)^3, (su)^3 \rangle$. It is an infinite Coxeter group because the element stu has infinite order. Then its poset of standard cosets is

$$P(W,S) = \Big\{ wW_\emptyset,\ wW_s,\ wW_t,\ wW_u,\ wW_{\{s,t\}},\ wW_{\{t,u\}},\ wW_{\{s,u\}} \ \Big|\ w \in W \Big\}$$

with order given by the reverse inclusion. Thus in $\Sigma(W,S)$ we have the vertex set V consisting of the cosets of the maximal standard subgroups, i.e. $W_{\{s,t\}}$, $W_{\{t,u\}}$ and $W_{\{s,u\}}$; the 1-skeleton is given by the left cosets of W_s, W_t and W_u and the 2-skeleton is in one-to-one correspondence with W itself. In Figure 2.2 we can see $\Sigma(W,S)$, which corresponds to the tiling of the Euclidean plane by infinite equilateral triangles.

Note that a Coxeter group W acts by left multiplication on the poset $P(W,S)$ and the action is by poset isomorphisms. Therefore, this induces an action of W on its Coxeter complex $\Sigma(W,S)$ by simplicial automorphisms. When $\Sigma(W,S)$ is a triangulation of a sphere, we say that the Coxeter group W is *spherical*, and this happens if and only if W is a finite Coxeter group. With the same spirit, Coxeter groups with an associated Coxeter complex which is a triangulation of an Euclidean space are called *Euclidean* and the Examples 2.39 and 2.40 show instances of this type of Coxeter groups.

Coxeter complexes are the main ingredient in the definition of buildings:

Definition 2.41. A building is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ, which are called apartments, satisfying the following conditions:

1. Each apartment Σ is a Coxeter complex;
2. For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them;
3. If Σ and Σ' are two apartments both containing A and B, then there is an isomorphism $\Sigma \longrightarrow \Sigma'$ fixing A and B pointwise.

During the years many different definitions for buildings have been given, and among these Δ could be a polyhedral complex with apartments being other geometric realizations of the Coxeter system (W, S), such as the Davis complex [AB08], or in the case that (W, S) is a geometric reflection group on $\mathbb{X}^n = \mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n$ we could take as apartments copies of $\mathbb{X}^n$ tiled by copies of the fundamental domain for the W-action on $\mathbb{X}^n$, which is a convex polytope. If apartments are spherical tilings we call the resulting buildings *spherical buildings*, while in the Euclidean context they are called *Euclidean buildings*.

As an immediate and fundamental example for the next chapter, note that every tree without valence-1 vertices is a Euclidean building with apartments isomorphic to the Coxeter complex of Example 2.39 and Figure 2.1.

Without going into deeper details we present the following theorem, which allows us to define a "nice" canonical Euclidean building $\mathcal{X}$ for a large family of algebraic groups $\mathcal{G}$ over p-adic fields. It was first proved for $\mathrm{SL}(2, \mathbb{Q}_p)$ by Goldman-Iwahori, then it was generalized to split groups by Iwahori-Matsumoto and finally to every reductive p-adic group by Bruhat-Tits [Mor13]. We won't need the theorem in its full generality, but only for $\mathcal{G} = \mathrm{SL}(n, \mathbb{Q}_p)$ and $\mathcal{G} = \mathrm{SO}(q; \mathbb{Q}_p)$ for q a non-degenerate quadratic form in n variables. Thus, for the sake of simplicity, we won't define the notion of reductive algebraic groups, and we will show the structure of $\mathcal{X}$ in these two cases only.

To understand the statement of Theorem 2.43 and in view of the next chapter, we introduce the following notation:

Definition 2.42. A *split torus* over $\mathbb{Q}_p$ is an algebraic group that is a direct product of finitely many copies of the multiplicative group of the field, $\mathbb{Q}_p^*$.

Split tori help us to identify apartments in $\mathcal{X}$ too, as we will see later on. The above mentioned theorem is the following:

Theorem 2.43 ([Tit79], [Mor13]). *Let $\mathcal{G}$ be a reductive group over $\mathbb{Q}_p$. There exists a canonical Euclidean building $\mathcal{X}$ called the Bruhat-Tits building of $\mathcal{G}$ such that:*

1. *$\mathcal{G}$ acts properly on $\mathcal{X}$ and the action is by simplicial automorphisms,*
2. *$\mathcal{X}$ is contractible,*
3. *$\mathcal{X}$ is finite-dimensional and $\dim \mathcal{X} = \mathrm{rank}_{\mathbb{Q}_p} \mathcal{G}$, where the $\mathbb{Q}_p$-rank of $\mathcal{G}$ is the dimension of a maximal split torus subgroup,*
4. *$\mathcal{X}$ is locally finite, i.e. each simplex is adjacent to only finitely many other simplices.*

From the canonicity of the Bruhat-Tits building associated to $\mathcal{G}$, a direct consequence is that:

Corollary 2.44. *If $\mathcal{H}$ and $\mathcal{G}$ are reductive groups over $\mathbb{Q}_p$ and $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then we have a natural inclusion of buildings $\mathcal{X} \hookrightarrow \mathcal{Y}$, where $\mathcal{X}$ is the building associated to $\mathcal{H}$ and $\mathcal{Y}$ is the one associated to $\mathcal{G}$.*

For what concerns our study, we have that:

Proposition 2.45 ([Bor91, §23])**.** *The subgroup of diagonal matrices in $\mathrm{SL}(n, \mathbb{Q}_p)$ is a split torus. Its dimension is $n-1$ and it is a maximal split torus in $\mathrm{SL}(n, \mathbb{Q}_p)$. Hence*

$$\mathrm{rank}_{\mathbb{Q}_p} \mathrm{SL}(n, \mathbb{Q}_p) = n - 1.$$

Moreover, if q is a non-degenerate quadratic form over $\mathbb{Q}_p$ with Witt index w, that is to say with maximal dimension of a fully degenerate subspace equal to w, then it holds that

$$\mathrm{rank}_{\mathbb{Q}_p} \mathrm{O}(q; \mathbb{Q}_p) = w.$$

In this case, if considering a basis of $\mathbb{Q}_p^n$ such that the quadratic form q is represented by the matrix

$$\begin{pmatrix} \begin{array}{cc|} 0 & 1 \\ 1 & 0 \\ \hline \end{array} & & & \\ & \ddots & & \\ & & \begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} & \\ & & & \begin{array}{|c} \\ \pm\mathrm{Id} \\ \\ \end{array} \end{pmatrix}$$

which has a total of w blocks of order 2×2 of signature zero and then a $\pm\mathrm{Id}$ block of order and sign depending on the signature of q, a maximal split torus S is the subgroup of $\mathrm{O}(q; \mathbb{Q}_p)$ consisting of diagonal matrices, that is:

$$S = \left\{ \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \ldots, \lambda_w, \lambda_w^{-1}, 1, \ldots, 1) \;\middle|\; \lambda_1, \ldots, \lambda_w \in \mathbb{Q}_p^* \right\}.$$

An important property of the action of the group $\mathcal{G}$ on its Bruhat-Tits building $\mathcal{X}$ is the following, which resembles the third condition of Definition 2.41:

Proposition 2.46 ([Tit79])**.** *If A' and A'' are two apartments in $\mathcal{X}$, there exists a $g \in \mathcal{G}$ such that $g \cdot A' = A''$ and fixes the intersection $A' \cap A''$ pointwise. In particular, this means that $\mathcal{G}$ acts transitively on the apartments of $\mathcal{X}$.*

It only remains to find an easy way to obtain apartments of the Bruhat-Tits building $\mathcal{X}$ knowing the group $\mathcal{G}$. It holds that:

Proposition 2.47 ([Tit79])**.** *Given a maximal split torus $S \leq \mathcal{G}$, there exists a unique apartment A such that $S \cdot A = A$, i.e. such that A is invariant under the action of the split torus S.*

2.6 Bruhat-Tits Building of $\mathrm{SL}(n, \mathbb{Q}_p)$

If $\mathcal{G} = \mathrm{SL}(n, \mathbb{Q}_p)$ we can define in an easy and direct way its Bruhat-Tits building. In the next chapter we will also need the building for $\mathrm{O}(q; \mathbb{Q}_p)$ where q will be a non-degenerate quadratic form in 5 variables of signature $(4, 1)$. From Proposition 2.45 we can then say that it is a 1-dimensional building, which is contractible by Theorem 2.43. Hence it is a tree, and it can be seen as a sub-building of the Bruhat-Tits building associated to $\mathrm{SL}(n, \mathbb{Q}_p)$.

Let $V = \mathbb{Q}_p^n$, an n-dimensional vector space over $\mathbb{Q}_p$. We recall that the valuation ring of $\mathbb{Q}_p$ is $\mathbb{Z}_p$ and its unique maximal ideal is generated by the prime number p. Moreover, the residue field $\mathbb{Z}_p/p\mathbb{Z}_p$ is isomorphic to the finite field with p elements, $\mathbb{F}_p$.

Definition 2.48. A $\mathbb{Z}_p$-*lattice* (or simply lattice) in $\mathbb{Q}_p^n$ is a $\mathbb{Z}_p$-submodule with n generators which spans the whole $\mathbb{Q}_p^n$. Equivalently, a $\mathbb{Z}_p$-submodule L of $\mathbb{Q}_p^n$ is a lattice if the natural map $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p^n$ given by $v \otimes q \longmapsto qv$ is an isomorphism.

Definition 2.49. Two lattices L_1 and L_2 are said to be *homothetic* if there exists $\lambda \in \mathbb{Q}_p^*$ such that $L_1 = \lambda L_2$, i.e. they differ by a non-zero scalar. Being homotetic is an equivalence relation; we will write $\mathcal{L} = [L]$ for the homothety (equivalence) class of the lattice L.

We now take as set of vertices of the Bruhat-Tits building $\mathcal{X}$ associated to $\mathrm{SL}(n, \mathbb{Q}_p)$ the set of homothety classes of lattices in $\mathbb{Q}_p^n$. Then we define 1-simplices as follows:

Definition 2.50. Two vertices $\mathcal{L}_1$ and $\mathcal{L}_2$ are adjacent if there are representatives $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_2$ such that

$$pL_2 \subsetneq L_1 \subsetneq L_2.$$

Note that the relation is symmetric: if $pL_2 \subsetneq L_1 \subsetneq L_2$ then it also holds that $pL_1 \subsetneq L_2' \subsetneq L_1$ for $L_2' = pL_2$, since $pL_1 \subsetneq pL_2 \subsetneq L_1 \subsetneq L_2$.

Moreover the following property holds:

Proposition 2.51. *If two vertices $\mathcal{L}$ and $\mathcal{L}'$ are adjacent then every pair of representatives L_1' and L_2' respectively of the first and second class have the property that either $L_1' \subsetneq L_2'$ or $L_2' \subsetneq L_1'$.*

Proof. The property follows from the fact that each $\lambda \in \mathbb{Q}_p^*$ can be factored as $\lambda = p^k u$ with $u \in \mathbb{Z}_p^*$ and $k \in \mathbb{Z}$; thus $\lambda L = p^k u L = p^k L$ and lattices in the same class as L can be characterized as

$$[L] = \left\{ p^k L \;\middle|\; k \in \mathbb{Z} \right\}.$$

In this way any two representatives differ from the preferred ones (i.e. the lattices for which $pL_2 \subsetneq L_1 \subsetneq L_2$) by powers of p, say $L_1' = p^h L_1$ and $L_2' = p^k L_2$.

If $h \geq k$ then, since $L_1 \subsetneq L_2$,

$$p^h L_1 \subsetneq p^h L_2 \subsetneq p^k L_2;$$

if instead $h < k$ then, since $pL_2 \subsetneq L_1$,

$$p^k L_2 = p^{k-1}(pL_2) \subsetneq p^{k-1} L_1 \subsetneq p^h L_1.$$

$\square$

To obtain the complete definition of the Bruhat-Tits building $\mathcal{X}$ it only remains to prescribe simplices of higher dimensions, and they are defined in the easiest way:

Definition 2.52. The Bruhat-Tits building of $\mathrm{SL}(n, \mathbb{Q}_p)$ is the flag complex having as vertices homothety classes of lattices in $\mathbb{Q}_p^n$ and edges given by the previous adjacency argument.

Thus, a set of $k+1$ vertices is a k-simplex if and only if the vertices are mutually adjacent.

We also give an explicit description of higher dimension simplices, which makes clear that $\dim \mathcal{X} = n - 1$.

Proposition 2.53. *If $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are mutually adjacent vertices then there exist (possibly after relabeling and reordering the homothety classes) representatives $L_1, \ldots, L_k$ respectively of $\mathcal{L}_1, \ldots, \mathcal{L}_k$ such that*

$$pL_k \subsetneq L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_k.$$

Proof. We prove this by induction. The base case is just the definition of edges in $\mathcal{X}$. For the induction step, suppose that we have $\mathcal{L}_1, \ldots, \mathcal{L}_{k+1}$ mutually adjacent vertices and that we already have $L_1, \ldots, L_k$ representatives such that

$$pL_k \subsetneq L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_k.$$

Now, since $\mathcal{L}_{k+1}$ is adjacent to $\mathcal{L}_k$, we can take L_{k+1} so that

$$pL_{k+1} \subsetneq L_k \subsetneq L_{k+1}.$$

If $pL_{k+1} \subsetneq L_1$ holds then we are done; if not, by the property we showed earlier, it necessarily holds that $L_1 \subsetneq pL_{k+1}$. Similarly, since $\mathcal{L}_{k+1}$ is adjacent to each other $\mathcal{L}_i$, we have $pL_{k+1} \subsetneq L_i$ or $L_i \subsetneq pL_{k+1}$.

Thus there exists a minimal index j such that $pL_{k+1} \subsetneq L_j$ and we have

$$pL_k \subsetneq L_1 \subsetneq \ldots \subsetneq L_{j-1} \subsetneq pL_{k+1} \subsetneq L_j \subsetneq \ldots \subsetneq L_k.$$

Replacing the representative L_{k+1} with the better representative pL_{k+1} we are done. Indeed, now we have $k+1$ representatives of $k+1$ mutually adjacent vertices that form a flag in the quotient $L_k/pL_k \cong \mathbb{F}_p^n$. Explicitly, if we define the lattices

$$M_i := \begin{cases} L_i & \text{if } 1 \leq i \leq j-1 \\ pL_{k+1} & \text{if } i = j \\ L_{i-1} & \text{if } j+1 \leq i \leq k+1 \end{cases}$$

which are $k+1$ representatives of the $k+1$ mutually incident vertices $\mathcal{L}_1, \dots, \mathcal{L}_{k+1}$, we have that

$$pM_{k+1} \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_k \subsetneq M_{k+1},$$

which proves the induction step. □

As an immediate consequence we have:

Corollary 2.54. *The maximal length of a chain of lattices is equal to n and each quotient of consecutive lattices in the chain is a 1-dimensional $\mathbb{F}_p$-vector space, i.e. $L_{i+1}/L_i \cong \mathbb{F}_p$.*

Hence, the dimension of maximal simplices in $\mathcal{X}$ is equal to $n-1$.

Proof. Taking a vertex $\mathcal{L}$ and a representative L, the quotient L/pL is isomorphic to an n-dimensional vector space over $\mathbb{F}_p$, and using the previous proposition we have that vertices contained in a common simplex give a flag of subspaces in $\mathbb{F}_p^n$, hence the thesis. □

The action of $\mathrm{SL}(n, \mathbb{Q}_p)$ on $\mathcal{X}$ is the simplicial action induced by

$$\mathrm{SL}(n, \mathbb{Q}_p) \ni M \colon [L] \longmapsto [ML].$$

This action extends naturally to $\mathrm{GL}(n, \mathbb{Q}_p)$ and since scalar matrices act trivially the action also descends to $\mathrm{PGL}(n, \mathbb{Q}_p)$. That's the reason why the building $\mathcal{X}$ is sometimes called *the building of* $\mathrm{PGL}(n, \mathbb{Q}_p)$ with a little abuse of notation.

The special linear group $\mathrm{SL}(n, \mathbb{Q}_p)$ doesn't act transitively on the set of vertices, while $\mathrm{GL}(n, \mathbb{Q}_p)$ does. Thus sometimes it is more useful to work with the Bruhat-Tits building $\mathcal{X}$ together with the simplicial action of the group $\mathrm{PGL}(n, \mathbb{Q}_p)$.

Buildings are difficult to visualize in general, but the Bruhat-Tits building of a rank-1 reductive algebraic group is a 1-dimensional contractible simplicial complex, hence a tree. It is easy to prove that:

Theorem 2.55. *The Bruhat-Tits building of* $\mathrm{SL}(2, \mathbb{Q}_p)$ *is a regular tree of degree $p+1$, i.e. every vertex has valence $p+1$.*

Proof. If L is a $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$, we claim that adjacent vertices are in one to one correspondence with non trivial proper vector subspaces of $\mathbb{F}_p^2$. For if L' is a lattice such that $pL \subsetneq L' \subsetneq L$, then the quotient L'/pL is a non trivial proper subspace of $L/pL \cong \mathbb{Z}_p^2/p\mathbb{Z}_p^2 \cong \mathbb{F}_p^2$. Conversely, any subspace of $\mathbb{F}_p^2$ can be seen through the isomorphism with L/pL as an intermediate lattice between pL and L.

Hence, the valence of the vertex $\mathcal{L} = [L]$ equals the number of 1-dimensional subspaces of $\mathbb{F}_p^2$, i.e. the number of lines in $\mathbb{F}_p^2$. In turn, lines in a vector space are exactly as many as points in the associated projective space. This computation is now straightforward:

$$\mathbb{P}^1(\mathbb{F}_p^2) = \{[1,0],[1,1],\dots,[1,p-1],[0,1]\}$$

implies that $\mathcal{L}$ has valence $p+1$. □

We can also characterize the vertices at distance m from v_0 in the Bruhat-Tits tree $\mathcal{T}$ for $\mathrm{SL}(2, \mathbb{Q}_p)$, where the distance considered is the natural one defined on graphs, i.e. the number of edges of a shortest path connecting the two vertices. With the same spirit of the previous proof, we have:

Proposition 2.56 ([Ser80])**.** *Each vertex of $\mathcal{T}$ is represented by a unique lattice $L \subseteq L_0$, where $L_0 = \langle e_1, e_2 \rangle$ is the standard lattice and $\langle - \rangle$ stands for the $\mathbb{Z}_p$-span, such that $L_0/L \cong \mathbb{Z}_p/p^m\mathbb{Z}_p$, where $m = d([L], [L_0])$ is the distance between the two vertices. The $\mathbb{Z}_p/p^m\mathbb{Z}_p$-module L_0/p^mL_0 is free of rank 2, and L/p^mL_0 is a direct factor of rank 1. Thus, vertices of $\mathcal{T}$ at distance m from $[L_0]$ correspond bijectively to points of the projective line*

$$\mathbb{P}(L_0/p^mL_0) \cong \mathbb{P}^1(\mathbb{Z}_p/p^m\mathbb{Z}_p)$$

where the space $\mathbb{P}^1(\mathbb{Z}_p/p^m\mathbb{Z}_p)$ consists of all pairs (x, y) of elements in $\mathbb{Z}_p/p^m\mathbb{Z}_p$ such that at least one of them is a unit, modulo scalar multiplication by units of $\mathbb{Z}_p$.

Let us return now to the general setting of the Bruhat-Tits building $\mathcal{X}$ associated to the group $\mathrm{SL}(n, \mathbb{Q}_p)$. Using repeatedly Definition 2.50 we see that:

Proposition 2.57. *Let $\mathcal{X}$ be the Bruhat-Tits building for $\mathrm{SL}(n, \mathbb{Q}_p)$. If $\mathcal{L}$ and $\mathcal{L}'$ are vertices at distance m, there are representatives L and L' of the two homothety classes such that $p^mL \subsetneq L' \subsetneq L$.*

Proof. The proof is an easy application of Definition 2.50. Let $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{L}_m = \mathcal{L}'$ and $\mathcal{L}_i$ be the vertices lying in a minimal-length path connecting $\mathcal{L}$ and $\mathcal{L}'$. Then there are representatives L_i for every $0 \leq i \leq m$ such that

$$\begin{gathered} pL_0 \subsetneq L_1 \subsetneq L_0 \\ pL_1 \subsetneq L_2 \subsetneq L_1 \\ \dots \\ pL_{m-1} \subsetneq L_m \subsetneq L_{m-1}. \end{gathered}$$

Putting these containments together we obtain

$$p^mL_0 \subsetneq p^{m-1}L_1 \subsetneq \ldots \subsetneq pL_{m-1} \subsetneq L_m \subsetneq \ldots \subsetneq L_1 \subsetneq L_0,$$

which implies that $p^mL_0 \subsetneq L_m \subsetneq L_0$. $\square$

Let $L_0 = \mathbb{Z}_p^n$ be the standard lattice in $\mathbb{Q}_p^n$. We treat its homothety class $[L_0]$ as a root, that is a preferred vertex from which in a certain sense the whole tree propagates. We now briefly study some subgroups of $\mathrm{PGL}(n, \mathbb{Q}_p)$ which stabilize a subset of vertices, such as for example $[L_0]$. In order to find the stabilizer of any vertex $\mathcal{L}$, it then suffices to conjugate the one we will now calculate with an element in $\mathrm{PGL}(n, \mathbb{Q}_p)$ mapping $[L_0]$ to $\mathcal{L}$. In this sense the tree $\mathcal{T}$ is homogeneous.

The computation will be an easy consequence of the following general decomposition:

Proposition 2.58 (Cartan decomposition [Tit79]). *The group* $\mathrm{GL}(n, \mathbb{Q}_p)$ *can be decomposed as*

$$\mathrm{GL}(n, \mathbb{Q}_p) = \mathrm{GL}(n, \mathbb{Z}_p)\, \mathrm{Z}\, \mathrm{GL}(n, \mathbb{Z}_p),$$

where

$$\mathrm{Z} = \left\{ \begin{pmatrix} p^{r_1} & & 0 \\ & \ddots & \\ 0 & & p^{r_n} \end{pmatrix} \;\middle|\; r_1, \ldots, r_n \in \mathbb{Z} \right\}.$$

It is easy now to prove that:

Proposition 2.59. *The stabilizer of the base vertex* $[L_0]$ *with respect to the action of* $\mathrm{PGL}(2, \mathbb{Q}_p)$*, where* $L_0 = \mathbb{Z}_p^2$ *is the standard lattice, is the subgroup of matrices with a representative with* $\mathbb{Z}_p$*-entries, i.e.*

$$\mathrm{Stab}_{\mathrm{PGL}(2,\mathbb{Q}_p)}([L_0]) = \mathrm{PGL}(2, \mathbb{Z}_p).$$

More generally, if $L_0 = \mathbb{Z}_p^n$*, the same holds for the action of* $\mathrm{PGL}(n, \mathbb{Q}_p)$*, i.e.*

$$\mathrm{Stab}_{\mathrm{PGL}(n,\mathbb{Q}_p)}([L_0]) = \mathrm{PGL}(n, \mathbb{Z}_p).$$

Proof. We first observe that elements in $\mathrm{GL}(n, \mathbb{Z}_p)$ stabilize the lattice $L_0 = \mathbb{Z}_p^n$ and hence the vertex $[L_0]$. Hence, if $M \in \mathrm{GL}(n, \mathbb{Q}_p)$ stabilizes the vertex $[L_0]$, we use the Cartan decomposition to write $M = N_1 D N_2$. Then there exists a $\lambda \in \mathbb{Q}_p^*$ such that

$$\lambda L_0 = M L_0 = N_1 D N_2 L_0 = N_1 D L_0$$

which is equivalent to

$$\lambda L_0 = \lambda N_1^{-1} L_0 = D L_0 = \begin{pmatrix} p^{r_1} & & \\ & \ddots & \\ & & p^{r_n} \end{pmatrix} L_0.$$

From this equality it follows that necessarily $r_1 = \ldots = r_n = r$ and that $M = p^r N_1 N_2$, i.e. M is the product of a scalar matrix with an element in $\mathrm{GL}(n, \mathbb{Z}_p)$. □

In order to study the (pointwise) stabilizer of a bigger set of vertices, we just need to intersect each vertex stabilizer. But if we consider the set of all those vertices at distance less or equal than m from $[L_0]$, a very nice connection with congruence subgroups appear.

We start with observing that elements in $\mathrm{GL}(n, \mathbb{Z}_p)$ stabilize not only the vertex $[L_0]$, but more specifically they stabilize the lattice L_0. Hence, if we consider the quotient L_0/pL_0, the action of $\mathrm{GL}(n, \mathbb{Z}_p)$ descends naturally to this space, which is an n-dimensional vector space over the field with p elements $\mathbb{F}_p$, as we saw for example in Theorem 2.55 for $n = 2$. Since in this quotient we killed every p-multiple in $\mathbb{Z}_p$, the action of $\mathrm{GL}(n, \mathbb{Z}_p)$ can also be seen by reducing each matrix entry modulo p. Recalling again, from Definition 2.50, that vertices which are adjacent to $[L_0]$ project to subspaces in L_0/pL_0, and the arguments in the proof of Theorem 2.55 can be easily generalized in dimension n by replacing $\mathbb{P}^1(\mathbb{F}_p^2)$ with the set of all proper non-trivial subspaces of $\mathbb{F}_p^n$, we obtain that an element $M \in \mathrm{GL}(n, \mathbb{Z}_p)$ fixes $[L_0]$ and all its neighbors if and only if acts as the identity on the quotient space L_0/pL_0. This in turn is equivalent to:

Proposition 2.60. *The stabilizer of the set of vertices at distance less or equal than 1 from $[L_0]$ in the Bruhat-Tits building $\mathcal{X}$ for $\mathrm{SL}(n, \mathbb{Q}_p)$ is the p-congruence subgroup $\Gamma(p) < \mathrm{GL}(n, \mathbb{Z}_p)$, i.e. the subgroup*

$$\Gamma(p) = \{M \in \mathrm{GL}(n, \mathbb{Z}_p) \mid M \equiv I \pmod{p}\}.$$

In the very same way, having in mind the generalization presented in Proposition 2.57, we can prove that:

Proposition 2.61. *The stabilizer of the set of vertices at distance less or equal than m from $[L_0]$ in the Bruhat-Tits building for $\mathrm{SL}(n, \mathbb{Q}_p)$ is the congruence subgroup $\Gamma(p^m) < \mathrm{GL}(n, \mathbb{Z}_p)$ of level p^m, i.e. the subgroup*

$$\Gamma(p^m) = \{M \in \mathrm{GL}(n, \mathbb{Z}_p) \mid M \equiv I \pmod{p^m}\}.$$

Proof. Using the same argument as above, by passing the action to the quotient L_0/p^mL_0 and reducing coefficients of matrices M in $\mathrm{GL}(n, \mathbb{Z}_p)$ modulo p^m, we obtain that vertices at distance $\leq m$ from $[L_0]$ are projected to non-trivial proper subspaces of L_0/p^mL_0. Hence they are fixed pointwise if and only if $M \equiv I \pmod{p^m}$. □

Chapter 3

Congruence RFRS Towers

In this chapter, we introduce the notion of Γ being (virtually) RFRS, where Γ is a finitely generated group. We then describe a criterion for Γ to admit a RFRS tower that consists entirely of congruence subgroups. After that, two examples are shown in which we produce a RFRS tower. The first one shows that principal congruence arithmetic link complements are RFRS, and the second is about Bianchi groups being virtually RFRS. We follow [AS19] throughout the chapter.

We recall that:

Definition 3.1. The Bianchi groups are the groups $\mathrm{PSL}(2, \mathcal{O}_d)$ where $\mathcal{O}_d$ is the ring of algebraic integers of $k = \mathbb{Q}(\sqrt{-d})$ and d is a square-free positive integer.

The main result will be the following:

Theorem 3.2. *The Bianchi groups* $\mathrm{PSL}(2, \mathcal{O}_d)$ *with* $d \not\equiv -1 \pmod 8$ *and* d *positive and square-free contain a RFRS tower consisting entirely of congruence subgroups. In particular, these Bianchi orbifolds virtually fiber on a congruence cover.*

3.1 RFRS towers

Let Γ be a finitely generated group with commutator subgroup $\Gamma^{(1)} = [\Gamma, \Gamma]$ and abelianization

$$\Gamma^{ab} = \Gamma/\Gamma^{(1)} \cong H_1(\Gamma; \mathbb{Z}).$$

With the same spirit we define also the *rational abelianization* and the *rational commutator subgroup*:

Definition 3.3. The rational abelianization of Γ is the image Γ^{rab} of Γ^{ab} in

$$\Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_1(\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_1(\Gamma; \mathbb{Q})$$

under the natural homomorphism $\Gamma^{ab} = \Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$.

Definition 3.4. The rational commutator subgroup $\Gamma_r^{(1)}$ of Γ is the kernel of the natural map $\Gamma \longrightarrow \Gamma^{rab}$ defined as before, i.e.

$$\Gamma_r^{(1)} = \mathrm{Ker}\left(\Gamma \longrightarrow \Gamma^{rab}\right).$$

We first observe that $\Gamma^{(1)} \leq \Gamma_r^{(1)}$ and that $\Gamma^{rab} \cong H_1(\Gamma;\mathbb{Z})/\mathrm{Torsion}$, so $\Gamma^{(1)}$ is a finite index subgroup of $\Gamma_r^{(1)}$.

We finally define what a RFRS tower is:

Definition 3.5. Given a finitely generated group Γ, let $\{\Gamma_j\}_{j\in\mathbb{N}}$ be a cofinal tower of finite index subgroups of Γ with $\Gamma_0 = \Gamma$. Explicitly, $\{\Gamma_j\}_{j\in\mathbb{N}}$ is such that

1. $\bigcap_{j\in\mathbb{N}} \Gamma_j = \{1\}$;
2. Γ_j is a finite index subgroup of Γ;
3. $\Gamma_{j+1} \leq \Gamma_j$ for all $j \geq 0$.

We say that $\{\Gamma_j\}_{j\in\mathbb{N}}$ is a *RFRS tower* if, in addition,

4. $(\Gamma_j)_r^{(1)} \leq \Gamma_{j+1}$ for all $j \geq 0$.

We say that Γ is RFRS if it admits such a tower and that is *virtually RFRS* if it contains a finite index subgroup that is RFRS.

Note that if there is a RFRS tower, then there is also a normal RFRS tower by passing to core subgroups (the largest normal refinement): the cofinal tower property is trivial; the finite index property is also trivial since the core of a finite index subgroup is again of finite index; finally, the RFRS condition follows because for g in Γ, we have $g\,(\Gamma_j)_r^{(1)}\,g^{-1} = \left(g\Gamma_j g^{-1}\right)_r^{(1)}$, so

$$\mathrm{Core}\,(\Gamma_{j+1}) = \bigcap_{g\in\Gamma} g\Gamma_{j+1}g^{-1} \geq \bigcap_{g\in\Gamma} g\,(\Gamma_j)_r^{(1)}\,g^{-1} = \bigcap_{g\in\Gamma}\left(g\Gamma_j g^{-1}\right)_r^{(1)} \geq (\mathrm{Core}(\Gamma_j))_r^{(1)}.$$

Thus the sequence of normal subgroups $\{\mathrm{Core}\,(\Gamma_j)\}_{j\in\mathbb{N}}$ is also a RFRS tower.

We finally remark that if Γ is RFRS, then also every subgroup H of Γ is.

The term RFRS stands for "residually finite rationally solvable", which means that for every element γ in Γ there exists a finite index normal subgroup $H \triangleleft \Gamma$ which does not contain γ and such that Γ/H is rationally solvable. The latter property generalizes the usual notion of solvable group, being a group G having a subnormal series $G = G_0 \triangleright \ldots \triangleright G_n = \{1\}$ whose quotient groups G_i/G_{i+1} are abelian (i.e. quotients of $H_1(G_i;\mathbb{Z}) = G_i^{ab}$). In this case the attribute *rational* specifies that the quotient groups have to be quotients of G_i^{rab}, the rational abelianization. It can be proved that the two definitions are actually equivalent:

Proposition 3.6. *A finitely generated group Γ is RFRS with respect to Definition 3.5 if and only if it is residually finite rationally solvable.*

Proof. Given a RFRS tower $\{\Gamma_j\}_{j\in\mathbb{N}}$ where, without loss of generality, each Γ_i is a normal subgroup of Γ, let $\gamma \in \Gamma$ be any element of the group. By the cofinal property there exists an index j such that $\gamma \notin \Gamma_j$. The subgroup Γ_j is of finite index by hypothesis and the quotient Γ/Γ_j is rationally solvable since

$$\Gamma/\Gamma_j \geq \Gamma_1/\Gamma_j \geq \ldots \geq \Gamma_j/\Gamma_j = \{1\}$$

is a subnormal series whose consecutive quotients

$$(\Gamma_i/\Gamma_j)\,/\,(\Gamma_{i+1}/\Gamma_j) \cong \Gamma_i/\Gamma_{i+1}$$

are quotients of $(\Gamma_i)^{rab}$ by hypothesis. This proves that if it exists a RFRS tower for Γ then Γ is residually finite rationally solvable.

Conversely, if Γ is residually finite rationally solvable, we define a RFRS tower inductively in the following way. We start by choosing any element $\gamma \in \Gamma$ in the group and taking a corresponding subgroup H for the property, i.e. such that $\gamma \notin H$, H is a finite index normal subgroup of Γ, and Γ/H is rationally solvable. Explicitly, if

$$\Gamma/H \geq \Gamma_1/H \geq \ldots \geq \Gamma_{n_1-1}/H \geq H/H = \{1\}$$

is the subnormal series having the property that Γ_i/Γ_{i+1} is a quotient of $(\Gamma_i)^{rab}$, it is then trivial that $(\Gamma_i)^{(1)}_r \leq \Gamma_{i+1}$. We now rename the subgroup H calling it Γ_{n_1}, and we repeat the same construction replacing Γ with Γ_{n_1}: we choose an element $\gamma_1 \in \Gamma_{n_1}$, we take a finite index normal subgroup H_1 not containing γ_1 and such that Γ_{n_1}/H_1 is rationally solvable. Again, the property ensures a subnormal series

$$\Gamma_{n_1}/H_1 \geq \Gamma_{n_1+1}/H_1 \geq \ldots \geq \Gamma_{n_2-1}/H_1 \geq H_1/H_1 = \{1\}$$

with the property $(\Gamma_i)^{(1)}_r \leq \Gamma_{i+1}$. Renaming H_1 and calling it Γ_{n_2}, we can iterate the inductive construction and define the tower $\{\Gamma_j\}_{j\in\mathbb{N}}$. The only thing to show in order to prove that this is a RFRS tower for Γ is that it is cofinal, being any other condition true by construction. But since Γ is finitely generated, it is countable; hence the inductive construction exhausts all the elements of the group, meaning that

$$\bigcap_{j\in\mathbb{N}} \Gamma_j = \{1\}\,.$$

□

Recall that if G is a group and $\Gamma \leq G$ a subgroup, the *commensurator* of Γ in G is the group consisting of those $g \in G$ such that $\Gamma \cap g\Gamma g^{-1}$ is of finite index in both Γ and $g\Gamma g^{-1}$.

Let us state and prove the next technical lemma, which is fundamental for the proof of Theorem 3.2:

Lemma 3.7. *Let G be a group and $\Gamma \leq G$ a finitely generated subgroup such that Γ^{ab} has no p-torsion for a prime number p. Suppose that $\{g_j\}_{j\in\mathbb{N}}$ is a sequence in G of elements in the commensurator of Γ in G with $g_0 = 1$. Defining*

$$\Delta_i = g_i \Gamma g_i^{-1}, \qquad \Gamma_n = \bigcap_{i=0}^{n} \Delta_i,$$

suppose that

1. *The subgroups $\{\Gamma_j\}_{j\in\mathbb{N}}$ form a cofinal tower of subgroups;*
2. *For each n, there exists some $0 \leq i \leq n-1$ such that $\Delta_i/(\Delta_i \cap \Delta_n)$ is an abelian p-group, i.e. each element has order p^k for some k.*

Then $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a RFRS sequence for Γ.

Proof. Since $\Delta_0 = \Gamma_0 = \Gamma$ and by hypotesis $\Delta_0/(\Delta_0 \cap \Delta_1) = \Gamma/\Gamma_1$ is an abelian p-group while Γ^{ab} has no p-torsion, the projection from Γ onto Γ/Γ_1 must factor not only through Γ^{ab}, but also through Γ^{rab}, so we have $(\Gamma_0)_r^{(1)} \leq \Gamma_1$.

We now show that $(\Gamma_n)_r^{(1)} \leq \Gamma_{n+1}$ also for all $n \geq 1$. We certainly have $(\Gamma_n)_r^{(1)} \leq \Gamma_n$, and since $\Gamma_{n+1} = \Gamma_n \cap \Delta_{n+1}$, to prove that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a RFRS sequence, we must prove that $(\Gamma_n)_r^{(1)} \leq \Delta_{n+1}$.

Fix $0 \leq i \leq n$ such that $\Delta_i/(\Delta_i \cap \Delta_n)$ is an abelian p-group. By definition we have $\Delta_i \cong \Gamma$, so Δ_i^{ab} has no p-torsion and hence by the very same argument as above $(\Delta_i)_r^{(1)} \leq (\Delta_i \cap \Delta_{n+1})$. To conclude the proof, note that $\Gamma_n \leq \Delta_i$ by construction, so looking at the definition of the rational commutator group we have

$$(\Gamma_n)_r^{(1)} \leq (\Delta_i)_r^{(1)} \leq (\Delta_i \cap \Delta_{n+1}).$$

This gives $(\Gamma_n)_r^{(1)} \leq \Delta_{n+1}$ as desired. □

3.2 $\mathfrak{p}$-congruence towers

Recall that if k is a number field, we denote with $\mathcal{O}_k$ the ring of integers. Let $\mathcal{G} \leq \mathrm{GL}(n,k)$ be a k-algebraic matrix group, and let $\Gamma = \mathcal{G}_{\mathcal{O}_k}$ its subgroup of $\mathcal{O}_k$-points. Given a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ and $j \geq 1$, we can consider $\Gamma(\mathfrak{p}^j)$ the *principal congruence subgroup of* Γ *of level* $\mathfrak{p}^j$, i.e. the set of all those elements that are congruent to the identity modulo $\mathfrak{p}^j$.

The sequence of subgroups $\{\Gamma(\mathfrak{p}^j)\}_{j\in\mathbb{N}}$ is called the $\mathfrak{p}$-*congruence tower.* It is a cofinal tower of finite index subgroups of Γ, since $\mathcal{O}_k$ is a Dedekind domain. Indeed, since each non-zero ideal has finite norm, we have that each $\Gamma(\mathfrak{p}^j)$ is a finite index subgroup of Γ; moreover, from the existence and uniqueness of factorization of ideals in Dedekind domain, it follows that the ideal

$$\bigcap_{j\in\mathbb{N}} \mathfrak{p}^j$$

is trivial and this ensures that

$$\bigcap_{j \in \mathbb{N}} \Gamma(\mathfrak{p}^j) = \{1\} .$$

So the only non trivial property that has to be verified for a $\mathfrak{p}$-congruence tower to be a RFRS tower is property (4) of Definition 3.5. Actually, the question of whether the $\mathfrak{p}$-congruence tower $\{\Gamma(\mathfrak{p}^j)\}_{j\in\mathbb{N}}$ is a RFRS tower is still open [AS19]. We should check for every j if $(\Gamma(\mathfrak{p}^j))_r^{(1)} \leq \Gamma(\mathfrak{p}^{j+1})$ or not. What we will produce are instead RFRS towers made of congruence groups, i.e. subgroups containing each a $\Gamma(\mathfrak{p}^k)$ for some k. In order to produce this we will use the Lemma 3.7 together with the following algebraic lemma.

Lemma 3.8. *Suppose k is a number field with algebraic integers $\mathcal{O}_k$, $\mathcal{G} \leq \mathrm{GL}(n,k)$ is a k-algebraic matrix group, and $\Gamma = \mathcal{G}_{\mathcal{O}_k}$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ and p the characteristic of the finite field $\mathcal{O}_k/\mathfrak{p}$. Then:*

1. *For all $j \geq 1$, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^{j+1})$ is an elementary abelian p-group, i.e. each non-trivial element has order p.*
2. *For all $h > j \geq 1$, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^h)$ is a p-group.*
3. *For all $j \geq 2$ and $h \leq 2j$, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^h)$ is abelian. In particular, $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^{j+2})$ is abelian.*

Proof. Suppose that $\mathfrak{p}$ is a principal ideal of $\mathcal{O}_k$ with generator π. If $\alpha \in \Gamma(\mathfrak{p}^j)$, then we can write

$$\alpha = \mathrm{Id} + \pi^j M$$

for some $M \in \mathrm{M}(n,n,\mathcal{O}_k)$. Then

$$\alpha^p = \sum_{h=0}^{p} \binom{p}{h} \pi^{jh} M^h = \mathrm{Id} + p\pi^j M + \pi^{j+1} N,$$

with $N \in \mathrm{M}(n,n,\mathcal{O}_k)$, which is congruent to the identity modulo π^{j+1}, since p is the characteristic of $\mathcal{O}_k/\mathfrak{p}$ and hence $p \in \mathfrak{p} = (\pi)$. This proves that every element of $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^{j+1})$ has order p.

Now, suppose that $\alpha,\ \beta \in \Gamma(\mathfrak{p}^j)$, and we write

$$\alpha = \mathrm{Id} + \pi^j M, \qquad \beta = \mathrm{Id} + \pi^j N.$$

Then

$$\alpha\beta = \left(\mathrm{Id} + \pi^j M\right)\left(\mathrm{Id} + \pi^j N\right) = \mathrm{Id} + \pi^j(M+N) + \pi^{2j} MN,$$

$$\beta\alpha = \left(\mathrm{Id} + \pi^j N\right)\left(\mathrm{Id} + \pi^j M\right) = \mathrm{Id} + \pi^j(M+N) + \pi^{2j} NM.$$

So α and β commute modulo π^h for all $h \leq 2j$. This completes the proof of the first assertion of the lemma and also proves the third one.

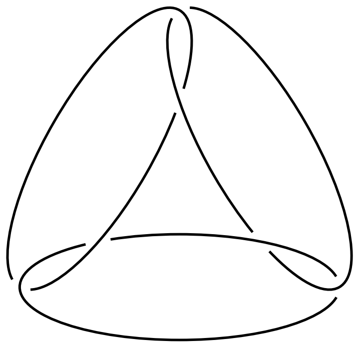

Figure 3.1: The 3-chain link 6^3_1.

The second statement is a consequence of the first and an easy induction on $h-j$. The base case $h=j+1$ is exactly the first statement of the Lemma and it has been already proved. For the induction step, we can write

$$\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^h) \cong \left(\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^{h+1})\right) / \left(\Gamma(\mathfrak{p}^h)/\Gamma(\mathfrak{p}^{h+1})\right).$$

Using both the inductive hypothesis and the base case, for which $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^h)$ and $\Gamma(\mathfrak{p}^h)/\Gamma(\mathfrak{p}^{h+1})$ are p-groups, we can conclude that $\Gamma(\mathfrak{p}^j)/\Gamma(\mathfrak{p}^{h+1})$ is a p-group too.

In the general setting, if $\mathfrak{p}$ is not a principal ideal, we consider $R(v_\mathfrak{p})$ the valuation ring of k with respect to its $\mathfrak{p}$-adic valuation, and we fix a uniformizing element π for $R(v_\mathfrak{p})$. We can now implement the same proof as above replacing $\mathcal{O}_k$ with $R(v_\mathfrak{p})$, since the residue fields $R(v_\mathfrak{p})/\pi R(v_\mathfrak{p})$ and $\mathcal{O}_k/\mathfrak{p}$ coincide. □

3.3 A first example: the magic manifold

We now describe an example that will be useful to explain the construction needed to prove the main result stated at the beginning of the chapter. We will prove that:

Theorem 3.9. *The fundamental group of the magic manifold admits a RFRS tower consisting entirely of congruence subgroups. In particular, the magic manifold is virtually fibered on a congruence cover.*

The magic manifold is the complement in S^3 of the 3-chain link 6^3_1 shown in Figure 3.1, and it can be geometrically constructed by gluing together two copies of a triangular prism. Moreover, it holds that its fundamental group is commensurable with the Bianchi group $\mathrm{PSL}(2,\mathcal{O}_7)$, being $\pi_1(S^3 \setminus 6^3_1)$ a subgroup of index 6 inside it [Thu80]. In the following we will sometimes use the notation $\mathrm{PGL}(2,\mathcal{O}_7)$ instead of the usual one for the Bianchi group $\mathrm{PSL}(2,\mathcal{O}_7)$, having in mind that these two objects are commensurable since the only invertible elements in $\mathcal{O}_7$ are ± 1.

More importantly, we can explicitly find that $\pi_1(S^3 \setminus 6^3_1) \cong \Gamma\left(\frac{1+\sqrt{-7}}{2}\right)$ is the principal congruence subgroup of level $\mathfrak{p} = \left(\frac{1+\sqrt{-7}}{2}\right)$. We will denote this group simply by Γ,

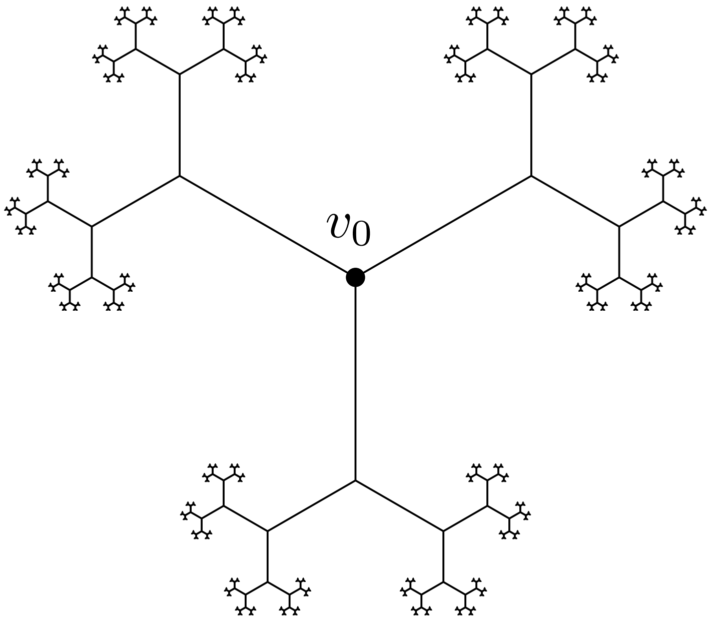

Figure 3.2: The Bruhat-Tits tree $\mathcal{T}$ associated to $\mathrm{SL}(2, \mathbb{Q}_2)$.

and we will consider it inside the larger group $\mathrm{PGL}(2, \mathbb{Q}(\sqrt{-7}))$.

Note that $\mathfrak{p}$ is a prime ideal with norm 2: indeed, recalling from Example 2.3 that $\mathcal{O}_7 = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$ and observing that $\frac{1+\sqrt{-7}}{2}\frac{1-\sqrt{-7}}{2} = 2 \in \mathfrak{p}$, we have that $\mathcal{O}_7/\mathfrak{p} \cong \mathbb{F}_2$. Equivalently, we can use Example 2.23 and Theorem 2.16 to see that the inertial degree of $\mathfrak{p}$ is $f = 1$. We will show that the magic manifold admits a 2-congruence tower that is RFRS.

Using the $\mathfrak{p}$-adic valuation, Corollary 2.17 implies that the completion of $\mathbb{Q}(\sqrt{-7})$ at $\mathfrak{p}$ is $\mathbb{Q}_2$, and hence we obtain an embedding i of $\mathrm{PGL}(2, \mathbb{Q}(\sqrt{-7}))$ into $\mathrm{PGL}(2, \mathbb{Q}_2)$.

The embedding can be explicitly seen using an observation made in the previous chapter: we only have to choose a root in $\mathbb{Q}_2$ of the polynomial $x^2 + 7$ and integrate it with the embedding of $\mathbb{Q}$ into $\mathbb{Q}_2$ using 2-adic expansions.

We already saw in Example 2.23 which are the roots of another polynomial closely related to the number field $\mathbb{Q}(\sqrt{-7})$, that is $x^2 - x + 2$, the minimal polynomial of the generator $\frac{1+\sqrt{-7}}{2}$ of the ring of integers $\mathcal{O}_7$. Using the root

$$x_1 = 2 + 2^2 + 2^5 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12} + 2^{15} + \dots$$

of $x^2 - x + 2$, we find that, since $(2x-1)^2 + 7 = 4(x^2 - x + 2)$, the element

$$y_1 = 2x_1 - 1 = 1 + 2 + 2^3 + 2^6 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12} + 2^{13} + 2^{16} + \dots$$

is a root of $x^2 + 7$ and hence we define the embedding i by mapping $\sqrt{-7}$ to y_1.

Consider the action of $\mathrm{PGL}(2, \mathbb{Q}_2)$ on its Bruhat-Tits tree $\mathcal{T}$, which as we have seen in the previous chapter is a 3-regular tree (see Figure 3.2). We recall that vertices of $\mathcal{T}$ are homothety classes of $\mathbb{Z}_2$-lattices in $\mathbb{Q}_2^2$, and two vertices $\mathcal{L}_1$ and $\mathcal{L}_2$ are adjacent if and only if they have representatives such that $L_2 \subsetneq L_1$ and $L_1/L_2 \cong \mathbb{F}_2$.

Let v_0 be the vertex associated with the standard lattice $\mathbb{Z}_2^2$. It is obvious that the ring of integers $\mathcal{O}_7$, which is contained in the valuation ring $R(v_\mathfrak{p})$, is mapped into the valuation ring $\mathbb{Z}_2$ of $\mathbb{Q}_2$ with respect to the 2-adic valuation. Moreover, we recall from Proposition 2.59 that the stabilizer of $v_0 = [\mathbb{Z}_2^2]$ is

$$\mathrm{Stab}_{\mathrm{PGL}(2,\mathbb{Q}_2)}(v_0) = \mathrm{PGL}(2, \mathbb{Z}_2).$$

Then it follows that $i(\Gamma) < \mathrm{PGL}(2, \mathbb{Z}_2)$ stabilizes the vertex v_0 and the element

$$g_1 = \begin{pmatrix} 0 & \frac{1+\sqrt{-7}}{2} \\ 1 & 0 \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{Q}(\sqrt{-7}))$$

is such that $i(g_1)$ exchanges v_0 with a neighbor v_1. Indeed, we have

$$i(g_1) = i \begin{pmatrix} 0 & \frac{1+\sqrt{-7}}{2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1 \\ 1 & 0 \end{pmatrix}$$

and if we denote with $\langle - \rangle$ the $\mathbb{Z}_2$-span and define

$$v_1 := \left[\left\langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \right]$$

we claim that

$$g_1 \cdot v_0 := i(g_1)(v_0) = \left[\left\langle \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \right] = v_1.$$

Henceforth we will drop the embedding i in our notations, using it implicitly when talking about the action of $\mathrm{PGL}(2, \mathbb{Q}(\sqrt{-7}))$ on the Bruhat-Tits tree $\mathcal{T}$.

The above equality is guaranteed by the fact that x_1 and 2 differ only by a unit in $\mathbb{Z}_2$, since the other root x_2 of $x^2 - x + 2$ is such that $x_1 x_2 = 2$ and, as already observed in Example 2.23, is invertible in $\mathbb{Z}_2$:

$$\mathbb{Z}_2^* \ni x_2 = 1 + 2 + 2^3 + 2^4 + 2^6 + 2^{13} + 2^{14} + \dots$$

Finally, it is trivial that $g_1 \cdot v_1 = v_0$. Notice also that g_1 lies in the commensurator of Γ, using that Γ is commensurable with $\mathrm{PGL}(2, \mathcal{O}_7)$ and Proposition 2.33.

We now check that

$$\Gamma(\mathfrak{p}^2) \leq \Gamma \cap g_1 \Gamma g_1^{-1},$$

and this allows us to apply Lemma 3.8 and to affirm that $\Gamma / \left(\Gamma \cap g_1 \Gamma g_1^{-1} \right)$ is an elementary abelian 2-group.

For, if $M \in \Gamma(\mathfrak{p}^2)$ we can write

$$M = \begin{pmatrix} 1 + a\pi^2 & b\pi^2 \\ c\pi^2 & 1 + d\pi^2 \end{pmatrix}, \qquad \pi = \frac{1 + \sqrt{-7}}{2}$$

where a, b, c, d are some elements of $\mathcal{O}_7$. Then we have

$$g_1^{-1} M g_1 = \begin{pmatrix} 0 & 1 \\ \pi^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 + a\pi^2 & b\pi^2 \\ c\pi^2 & 1 + d\pi^2 \end{pmatrix} \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 + d\pi^2 & c\pi^3 \\ b\pi & 1 + a\pi^2 \end{pmatrix}$$

which is clearly congruent to the identity matrix modulo $\mathfrak{p} = (\pi)$.

We present here another idea for the proof of the inclusion $\Gamma(\mathfrak{p}^2) \leq \Gamma \cap g_1 \Gamma g_1^{-1}$ and the fact that g_1 belongs to the commensurator of Γ, because it stresses an important property in terms of the final result of this section.

Our setting is $\mathrm{PGL}(2, \mathbb{Q}(\sqrt{-7}) < \mathrm{PGL}(2, \mathbb{Q}_2)$, and we recall that the embedding i maps $\frac{1+\sqrt{-7}}{2}$ to $x_1 = 2 + 2^2 + 2^5 + 2^7 + \ldots = 2(1 + 2 + 2^4 + 2^6 + \ldots)$. As already noted before, x_1 and 2 differ only by a unit in $\mathbb{Z}_2$, hence reducing modulo 2 the elements in $i(\mathrm{PGL}(2, \mathcal{O}_7))$ is the same as reducing modulo $\mathfrak{p}$ the elements in $\mathrm{PGL}(2, \mathcal{O}_7)$.

We proved in Proposition 2.61 that the congruence subgroup of level 2^m in $\mathrm{PGL}(2, \mathbb{Q}_2)$ is the stabilizer of the set of vertices at distance $\leq m$ from v_0. With the above observation we can also say that $\Gamma(\mathfrak{p}^m)$ is the stabilizer of every vertex at distance $\leq m$ from v_0 inside $\mathrm{PGL}(2, \mathcal{O}_7)$.

Hence an element in $\mathrm{PGL}(2, \mathcal{O}_7)$ belongs to $\Delta_0 := \Gamma = \Gamma(\mathfrak{p})$ if and only if it fixes the vertex v_0 and any of its neighbors (there are three of them). By conjugating with g_1, we also have that $\Delta_1 := g_1 \Gamma g_1^{-1}$ is the stabilizer of the vertex v_1 and its neighbors inside $g_1 \,\mathrm{PGL}(2, \mathcal{O}_7) g_1^{-1}$. But now, since g_1 has determinant equal to $-\pi$ and has entries in $\mathcal{O}_7$, by the Cramer's formula for the inverse matrix we have that both g_1 and πg_1^{-1} have entries in the ring of integers $\mathcal{O}_7$. Thus if we write any element $M \in \Gamma(\mathfrak{p}^2)$ as $I + \pi^2 N$ we obtain that

$$g_1^{-1}(I + \pi^2 N) g_1 = I + (\pi g_1^{-1})(\pi N) g_1 \in \Gamma(\mathfrak{p}) = \Gamma \leq \mathrm{PGL}(2, \mathcal{O}_7),$$

that is to say $\Gamma(\mathfrak{p}^2) \leq \Delta_1 = g_1 \Gamma g_1^{-1}$.

From this it follows that $\Delta_0 \cap \Delta_1$ contains a $\mathfrak{p}$-congruence subgroup and hence is of finite index in $\Gamma(\mathfrak{p})$. Similarly, we prove again that

$$g_1 \Gamma(\mathfrak{p}^2) g_1^{-1} \leq \Delta_0 \cap \Delta_1 \leq \Delta_1 = g_1 \Gamma g_1^{-1} = g_1 \Gamma(\mathfrak{p}) g_1^{-1}$$

since g_1 and πg_1^{-1} both have entries in $\mathcal{O}_7$. From this it follows again that $\Delta_0 \cap \Delta_1$ is of finite index in $\Delta_1 = g_1 \Gamma(\mathfrak{p}) g_1^{-1}$ and hence that g_1 belongs to the commensurator of Γ.

We now return to the inductive construction of a sequence $\{g_n\}_{n \in \mathbb{N}}$: let us define $g_n \in \mathrm{PGL}(2, \mathbb{Q}(\sqrt{-7}))$ and v_n in $\mathcal{T}$ by choosing some v_i for $0 \leq i \leq n-1$ for which not all neighbors of v_i are contained in $\{v_0, \dots, v_{n-1}\}$, letting v_n be one such neighbor of v_i, and taking g_n to be a conjugate of g_1 that switches v_i and v_n. Explicitly, let $h_n \in \mathrm{PGL}(2, \mathbb{Q}(\sqrt{-7}))$ such that $h_n \cdot v_0 = v_i$ and $h_n \cdot v_1 = v_n$. We define then $g_n := h_n g_1 h_n^{-1}$. Finally, let $\Delta_n := g_n \Delta_i g_n^{-1}$.

Our next claim is that $\Delta_i / (\Delta_i \cap \Delta_n)$ is an elementary abelian 2-group. To see this, we prove that it is isomorphic to $\Delta_0/(\Delta_0 \cap \Delta_1)$. Indeed, we have:

$$\begin{aligned}\Delta_i/(\Delta_i \cap \Delta_n) &= \Delta_i/(\Delta_i \cap g_n \Delta_i g_n^{-1}) = \Delta_i/(\Delta_i \cap h_n g_1 h_n^{-1} \Delta_i h_n g_1^{-1} h_n^{-1}) \\ &= h_n \left(h_n^{-1} \Delta_i h_n / (h_n^{-1} \Delta_i h_n \cap g_1 h_n^{-1} \Delta_i h_n g_1^{-1}) \right) h_n^{-1} \\ &\cong h_n^{-1} \Delta_i h_n / (h_n^{-1} \Delta_i h_n \cap g_1 h_n^{-1} \Delta_i h_n g_1^{-1}).\end{aligned}$$

Now, since each Δ_j is defined as a conjugate of an already defined Δ_k, and $\Delta_1 = g_1 \Gamma g_1^{-1}$, it follows that $h_n^{-1} \Delta_i h_n \cong \Gamma$ being a conjugate of it. Using this we finally obtain that

$$\Delta_i/(\Delta_i \cap \Delta_n) \cong \Gamma/(\Gamma \cap g_1 \Gamma g_1^{-1}) = \Delta_0/(\Delta_0 \cap \Delta_1).$$

Hence it is proved that $\Delta_i/(\Delta_i \cap \Delta_n)$ is an elementary abelian 2-group, being isomorphic to $\Delta_0/(\Delta_0 \cap \Delta_1)$.

Each g_n commensurates Γ for the same reason g_1 does, i.e. Proposition 2.33.

Let v_n vary over all vertices of $\mathcal{T}$. If we define

$$\Gamma_n = \bigcap_{i=0}^{n} \Delta_i,$$

we clearly have that $\bigcap_{n \in \mathbb{N}} \Gamma_n$ stabilizes each vertex of $\mathcal{T}$, that is to say every homothety class of lattices in $\mathbb{Q}_2^2$. Therefore we have $\bigcap_{n \in \mathbb{N}} \Gamma_n = \{\mathrm{Id}\}$, and $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a cofinal tower. In particular, by construction it satisfies all the hypotesis of Lemma 3.7, and hence it is a RFRS tower for Γ.

We stress that in our dissertation we have implicitly proved that each Γ_n is a congruence subgroup. The reason behind this is the same as the alternative proof of the commensurability of Δ_0 and Δ_1. Indeed, by the density of $\mathcal{O}_7$ in $\mathbb{Z}_2$ it is possible to define the element h_n of the inductive construction to be the product of an element in $\mathrm{PSL}(2, \mathcal{O}_7)$ and a diagonal matrix of the form $\mathrm{diag}(\pi^{n_1}, \pi^{n_2})$, using the Cartan decomposition of Proposition 2.58. Then, it holds that there exists a k such that both $\pi^k g_n$ and $\pi^k g_n^{-1}$ have entries in $\mathcal{O}_7$ and hence with a similar argument to the base case we have that every intersection $\Delta_0 \cap \Delta_n$ contains a congruence subgroup $\Gamma(\mathfrak{p}^{k_n})$ for some k_n.

Now, the remaining observation that has to be done in order to prove that Γ_n is a congruence subgroup is that the intersection of congruence subgroups is again a congruence subgroup: in this case it is sufficient to set

$$j_n := \max \{k_1 = 2, k_2, \dots, k_n\}$$

to have the inclusion

$$\Gamma(\mathfrak{p}^{j_n}) \leq (\Delta_0 \cap \Delta_1) \cap (\Delta_0 \cap \Delta_2) \cap \ldots \cap (\Delta_0 \cap \Delta_n) = \Delta_0 \cap \ldots \cap \Delta_n = \Gamma_n.$$

Thus we have showed a method to define a congruence RFRS tower, namely a RFRS tower consisting entirely of congruence subgroups for the fundamental group Γ of the magic manifold.

At this point, using Theorem 1.46 which links the theories of RFRS towers and of virtual fibering of manifolds, we proved Theorem 3.9.

We remark that the condition on Γ^{ab} not having 2-torsion is satisfied since link complements have torsion-free first homology group. Hence the very same construction works also for any other principal congruence arithmetic link. However, only a finite number of such links exist, and a complete list can be found in [BGR19].

More generally, this construction works for congruence subgroups of arithmetic Kleinian groups with no p-torsion in their first homology for p a prime number contained in the prime ideal defining the congruence subgroup.

Finally, observe that the fact that the magic manifold is virtually fibered is not new: indeed, the magic manifold is itself fibered (see for example [Lei02]). With this example we wanted to present in a simplified version the procedure that we will use again in next section in order to prove Theorem 3.2.

3.4 Bianchi groups and $\mathrm{O}(4,1;\mathbb{Z})$

Let $\mathrm{O}(4,1;\mathbb{Z})$ be the group of integral automorphisms of the quadratic form q_0, which is represented by the matrix

$$Q_0 = \mathrm{diag}(1,1,1,1,-1).$$

It will be convenient to change coordinates, so let $\alpha \in \mathrm{SL}_5(\mathbb{Z})$ be the matrix

$$\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{pmatrix}$$

and $\mathrm{O}(q;\mathbb{Z}) = \alpha^{-1}\,\mathrm{O}(4,1;\mathbb{Z})\alpha$, where q is the quadratic form having matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

With this change of coordinates, the congruence subgroups $\Gamma(N) < \mathrm{O}(4,1;\mathbb{Z})$ are sent in $\alpha^{-1}\Gamma(N)\alpha$, but since α is integral of determinant one, the latter is again the congruence subgroup of level N of $\mathrm{O}(q;\mathbb{Z})$. Thus, with a little abuse of notation, we continue calling it just $\Gamma(N)$.

In order to apply Lemma 3.7 and to construct a RFRS tower using the same approach as before, we need to find a principal congruence subgroup satisfying the torsion condition of that Lemma.

Theorem 3.10. *The abelianization* $\Gamma(2)^{ab}$ *of the level 2 congruence subgroup* $\Gamma(2)$ *inside* $\mathrm{O}(4,1;\mathbb{Z})$ *is a 2-torsion group. The level 4 congruence subgroup* $\Gamma(4)$*, instead, has torsion-free abelianization.*

Proof. If we consider $\Gamma(2)$, it is true that it is a right-angled Coxeter group [RT00, Theorem 7]. By computing its abelianization $\Gamma(2)^{ab} = \Gamma(2)/\Gamma(2)^{(1)}$, it is possible to show that it is an elementary abelian 2-group.

Moreover, it can be shown that the orders of the elementary abelian 2-groups $\Gamma(2)^{ab} = \Gamma(2)/\Gamma(2)^{(1)}$ and $\Gamma(2)/\Gamma(4)$ are equal, and since an elementary calculus shows that $\Gamma(2)^{(1)} \leq \Gamma(4)$, it follows that $\Gamma(2)^{(1)} = \Gamma(4)$. Then if we use [BP14, §4.5] to prove that the abelianization of $\Gamma(2)^{(1)}$ is isomorphic to the reduced degree-zero homology of a certain complex, we conclude that $\Gamma(4)^{ab}$ is of course torsion-free.

In a more practical and explicit way, one can use a computer algebra program like Magma to compute $\Gamma(4)$, giving a presentation of $\mathrm{O}(4,1;\mathbb{Z})$ as a Coxeter group, and obtaining that in fact $\Gamma(4)^{ab} \cong \mathbb{Z}^{55}$. In particular, it has no 2-torsion. □

Hence our study will be focused on defining a congruence RFRS tower for $\Gamma(4)$, since it has no 2-torsion, and this would prove that $\mathrm{O}(4,1;\mathbb{Z})$ is virtually RFRS and has a RFRS tower consisting entirely of congruence subgroups.

Since we want to work with $\Gamma(4) \leq \mathrm{O}(q;\mathbb{Q})$, we choose the prime ideal $\mathfrak{p} = (2)$ in $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$. As in the previous example, we consider $\mathbb{Q}_2$ the completion of $\mathbb{Q}$ with respect to the $\mathfrak{p}$-adic valuation, which is the 2-adic valuation. We will need to investigate on the structure of the Bruhat-Tits tree $\mathcal{T}$ associated to $\mathrm{O}(q;\mathbb{Q}_2)$.

The following dissertation aims to prove that:

Proposition 3.11. *The Bruhat-Tits building* $\mathcal{T}$ *associated to* $\mathrm{O}(q;\mathbb{Q}_2)$ *is a* $(5,3)$*-biregular tree as in Figure 3.3, i.e. there is a bipartition of the vertices into two subsets* U *and* W *such that each edge of* $\mathcal{T}$ *connects a vertex in* U *to one in* W*, such that every vertex in* U *has 5 adjacent vertices in* $\mathcal{T}$ *(and we say that the vertex has valence 5) and every vertex in* W *has valence 3 in* $\mathcal{T}$*.*

If we consider $\mathrm{O}(q;\mathbb{Q}_2)$ as a subgroup of $\mathrm{GL}(5,\mathbb{Q}_2)$, we obtain an injection of buildings $\mathcal{T} \hookrightarrow \mathcal{X}$, where $\mathcal{X}$ is the Bruhat-Tits building associated with $\mathrm{PGL}(5,\mathbb{Q}_2)$. We recall

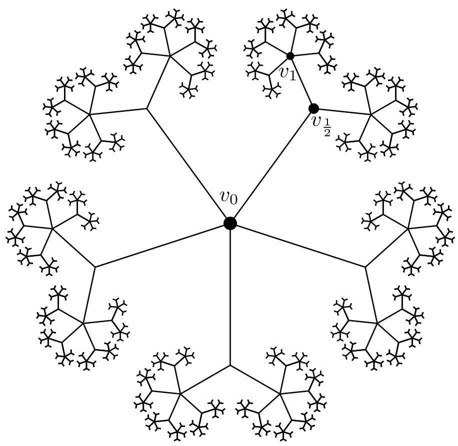

Figure 3.3: The Bruht-Tits tree $\mathcal{T}$ associated to $\mathrm{O}(q;\mathbb{Q}_2)$ is a $(5,3)$-biregular tree.

that $\mathcal{X}$ is a 4-dimensional building, while $\mathcal{T}$ is the building associated with the rank-1 group $\mathrm{O}(q;\mathbb{Q}_2)$ and hence it is a tree.

As in the example about the magic manifold, we fix the base vertex

$$v_0 := [\langle e_1,\ e_2,\ e_3,\ e_4,\ e_5\rangle]$$

where the e_i form the basis for which q has matrix Q and again $\langle - \rangle$ denotes the $\mathbb{Z}_2$-span. We recall again, from Proposition 2.59, that the stabilizer of v_0 in $\mathrm{GL}(5,\mathbb{Q}_2)$ is generated by $\mathrm{GL}(5,\mathbb{Z}_2)$ and the scalar matrices, since the vertex is defined as a lattice up to homothety. This implies that the subgroup of $\mathrm{GL}(5,\mathbb{Z}_2)$ of matrices which preserve the quadratic form q, namely $\mathrm{O}(q;\mathbb{Z}_2)$, is the stabilizer of v_0 inside $O(q;\mathbb{Q}_2)$. Moreover, $\mathrm{O}(q;\mathbb{Z}_2)$ stabilizes v_0 by fixing the standard representative $\mathbb{Z}_2^5$.

In general, the stabilizer of a vertex in $\mathcal{T}$ will be useful to understand the neighbors of that vertex, once at least one of them is known. Thus, the first thing to do is to explicitly find vertices belonging to an apartment.

Following what already said in the previous chapter, to find an apartment in $\mathcal{T}$ it suffices to take a maximal split torus in $\mathrm{O}(q;\mathbb{Q}_2)$. In this case, since $\mathcal{T}$ is 1-dimensional, apartments are lines, i.e. Coxeter complexes like the one in Example 2.39. Thus, we choose the 1-dimensional split torus S defined as

$$S := \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \;\middle|\; \lambda \in \mathbb{Q}_2^* \right\}$$

where Id is the 3×3 identity matrix matrix.

We then define the apartment A of $\mathcal{T}$ to be the invariant line with respect to the action of S.

We note that, choosing a unit $u \in \mathbb{Z}_2^*$, the matrix

$$N = \begin{pmatrix} u & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & u^{-1} \end{pmatrix}$$

where Id is the 3×3 identity matrix, belongs both to S and to $\mathrm{O}(q; \mathbb{Z}_2)$, so it stabilizes the vertex v_0 which is then contained in the S-invariant line.

Therefore we can see the apartment A as the convex hull of the S-orbit of the vertex v_0. This orbit can be easily understood, since elements $\lambda \in \mathbb{Q}_2^*$ can be factored as $\lambda = 2^r u$ with $r \in \mathbb{Z}$ and $u \in \mathbb{Z}_2^*$ and it is sufficient to see the action of the subgroup S' of S consisting of the matrices of the form

$$\begin{pmatrix} 2^r & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & 2^{-r} \end{pmatrix}$$

where Id is the 3×3 identity matrix and $r \in \mathbb{Z}$.

We thus obtain the family of vertices $\{v_r\}_{r \in \mathbb{Z}}$ where

$$v_r := [\langle 2^r e_1,\ e_2,\ e_3,\ e_4,\ 2^{-r} e_5 \rangle]$$

and S' acts on this family by translations.

It remains to investigate if these are all the vertices belonging to A or if there are more than the $\{v_r\}_{r \in \mathbb{Z}}$. To see this, we recall that we are studying the Bruhat-Tits tree $\mathcal{T}$ as a sub-building of the building associated to $\mathrm{PGL}(5, \mathbb{Q}_2)$, whose adjacent vertices are the ones for which a representative of one has index 2 inside a representative of the other. Proposition 2.51 helps us to prove that:

Lemma 3.12. *The vertices v_r and v_s are not adjacent for every choice of $r, s \in \mathbb{Z}$.*

Proof. If $r = s$ the thesis is obvious since a vertex is never adjacent to itself. Without loss of generality we can then suppose that $r > s$.

Let L_r and L_s be the following representatives of the vertices v_r and v_s:

$$\begin{aligned} L_r &:= \langle 2^r e_1,\ e_2,\ e_3,\ e_4,\ 2^{-r} e_5 \rangle\,, \\ L_s &:= \langle 2^s e_1,\ e_2,\ e_3,\ e_4,\ 2^{-s} e_5 \rangle\,. \end{aligned}$$

Since $r > s$ we have that $2^{-r} e_5 \notin L_s$ but at same time $2^s e_1 \notin L_r$. Thus it cannot hold that $L_r \subsetneq L_s$ or $L_s \subsetneq L_r$, contradicting Proposition 2.51. ☐

The previous proof suggests where to search for new vertices lying between two of the $\{v_r\}_r$. Indeed, if we define

$$\begin{aligned} L_{r+\frac{1}{2}} &:= \left\langle 2^{r+1} e_1,\ e_2,\ e_3,\ e_4,\ 2^{-r} e_5 \right\rangle, \\ v_{r+\frac{1}{2}} &:= \left[L_{r+\frac{1}{2}} \right], \end{aligned}$$

it is immediate to prove that $v_{r+\frac{1}{2}}$ is adjacent to v_r and v_{r+1}. Using the same notations as in the previous proof, we have

$$\left\langle 2^{r+1}e_1,\ 2e_2,\ 2e_3,\ 2e_4,\ 2^{-r+1}e_5 \right\rangle \subsetneq \left\langle 2^{r+1}e_1, e_2, e_3, e_4, 2^{-r}e_5 \right\rangle \subsetneq \left\langle 2^{r}e_1, e_2, e_3, e_4, 2^{-r}e_5 \right\rangle,$$

that is to say

$$2L_r \subsetneq L_{r+\frac{1}{2}} \subsetneq L_r,$$

and

$$\left\langle 2^{r+2}e_1, 2e_2, 2e_3, 2e_4, 2^{-r}e_5 \right\rangle \subsetneq \left\langle 2^{r+1}e_1, e_2, e_3, e_4, 2^{-r}e_5 \right\rangle \subsetneq \left\langle 2^{r+1}e_1, e_2, e_3, e_4, 2^{-r-1}e_5 \right\rangle,$$

which is

$$2L_{r+1} \subsetneq L_{r+\frac{1}{2}} \subsetneq L_{r+1}.$$

Recalling that A is a 1-dimensional Euclidean Coxeter complex, and hence a line, we have proved that the apartment A is the line with vertex set $\{v_\alpha \mid \alpha \in \frac{1}{2}\mathbb{Z}\}$ and in which v_α is adjacent to v_β if and only if $|\alpha - \beta| = \frac{1}{2}$.

At this point the work is almost done; since $\mathrm{O}(q;\mathbb{Q}_2)$ acts transitively on apartments of $\mathcal{T}$ (see Proposition 2.46) and the apartment A is the S'-orbit of $\{v_0, v_{\frac{1}{2}}\}$, we have proved that:

Proposition 3.13. *The vertex set of $\mathcal{T}$ is the $\mathrm{O}(q;\mathbb{Q}_2)$-orbit of $\{v_0, v_{\frac{1}{2}}\}$.*

In order to complete the proof of Proposition 3.11 it only remains to study the valence of the two vertices v_0 and $v_{\frac{1}{2}}$; then using the simplicial action of $\mathrm{O}(q;\mathbb{Q}_2)$ on $\mathcal{T}$ and Proposition 3.13 we obtain the bipartition $U = \mathrm{O}(q;\mathbb{Q}_2)\cdot v_0$ and $W = \mathrm{O}(q;\mathbb{Q}_2)\cdot v_{\frac{1}{2}}$ needed.

Lemma 3.14. *The vertex $v_0 \in \mathcal{T}$ has valence 5.*

Lemma 3.15. *The vertex $v_{\frac{1}{2}} \in \mathcal{T}$ has valence 3.*

Proof of Lemma 3.14. We already noted a few facts that are going to be useful: first of all, the group $\mathrm{O}(q;\mathbb{Q}_2)$ acts transitively on apartments and the action is simplicial; then, the stabilizer of the vertex v_0 is generated by the subgroup $\mathrm{O}(q;\mathbb{Z}_2)$ and scalar matrices; finally, the subgroup S' of the split torus S acts on the apartment A by translations. Moreover, observe that the matrix Q acts on A, too. It is orthogonal since $Q^TQQ = Q^3 = Q$ and it belongs to $\mathrm{O}(q;\mathbb{Z}_2)$; therefore it fixes the lattice L_0 and hence the vertex v_0, and it is easy to see that it acts as a reflection of the line A with respect to v_0. Indeed, taking as representatives of $v_{\frac{1}{2}}$ and $v_{-\frac{1}{2}}$ the usual lattices

$$L_{\frac{1}{2}} := \langle 2e_1,\ e_2,\ e_3,\ e_4,\ e_5 \rangle,$$
$$L_{-\frac{1}{2}} := \langle e_1,\ e_2,\ e_3,\ e_4,\ 2e_5 \rangle$$

we have that $QL_{\frac{1}{2}} = L_{-\frac{1}{2}}$ and $QL_{-\frac{1}{2}} = L_{\frac{1}{2}}$.

Having in mind these observations, we now claim that it is sufficient to compute the $\mathrm{O}(q;\mathbb{Z}_2)$-orbit of $v_{\frac{1}{2}}$ to obtain all the neighbors of v_0. In fact, if y is a vertex in $\mathcal{T}$ adjacent to v_0, by transitivity there exists an element $M \in \mathrm{O}(q;\mathbb{Q}_2)$ mapping v_0 and y inside the apartment A. By Proposition 2.46 we can suppose that v_0 is fixed by M and hence that $M \in \mathrm{O}(q;\mathbb{Z}_2)$; finally, possibly after post-composing with Q we can also suppose without loss of generality that y is mapped to $v_{\frac{1}{2}}$. Hence every neighbor of v_0 is in the same $\mathrm{O}(q;\mathbb{Z}_2)$-orbit of $v_{\frac{1}{2}}$, and the claim is proved.

Recall now that given the representative L_0 of v_0, neighbors of v_0 in $\mathcal{X}$ are in one-to-one correspondence with proper non-zero subspaces of

$$V_0 := L_0/2L_0 \cong \mathbb{F}_2^5.$$

Let $\{\overline{e_i}\}_{i\leq 5}$ be the basis of V_0 induced by the basis elements $e_1,\dots,e_5$ in $\mathbb{Q}_2^5$. The image of $L_{\frac{1}{2}}$ in V_0 is the subspace $\langle \overline{e_2},\ \overline{e_3},\ \overline{e_4},\ \overline{e_5}\rangle$, which can be identified with the orthogonal complement $\overline{e_5}^{\perp}$ of $\overline{e_5}$ with respect to the quadratic form $\overline{q}_0$ on V_0 induced by the restriction of q to L_0. Since $\mathrm{O}(q;\mathbb{Z}_2)$ preserves the lattice L_0, it acts on V_0 and by passing the action to the quotient we can reduce the entries of the matrices of the group modulo 2.

To prove the lemma it then suffices to compute the orbit of $\overline{e_5}^{\perp}$ with respect to the action of the image of $\mathrm{O}(q;\mathbb{Z}_2)$ under the reduction modulo 2, and since $v^{\perp} = \overline{e_5}^{\perp}$ if and only if $v = \overline{e_5}$, we only have to compute the orbit of the vector $\overline{e_5}$.

It is possible to check that there are 5 elements in its orbit and hence that the neighbors of v_0 in $\mathcal{T}$ have the following representatives:

$$\begin{gathered}
\langle e_1,\ e_2,\ e_3,\ e_4,\ 2e_5\rangle\\
\langle 2e_1,\ e_2,\ e_3,\ e_4,\ e_5\rangle\\
\langle 2e_1,\ e_1+e_2,\ e_1+e_3,\ e_4,\ e_1+e_2+e_3+e_5\rangle\\
\langle 2e_1,\ e_1+e_2,\ e_3,\ e_1+e_4,\ e_1+e_2+e_4+e_5\rangle\\
\langle 2e_1,\ e_2,\ e_1+e_3,\ e_1+e_4,\ e_1+e_3+e_4+e_5\rangle.
\end{gathered}$$

□

Proof of Lemma 3.15. The proof is very similar to the proof of Lemma 3.14. Here, instead of the matrix Q, we use the matrix

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{O}(q;\mathbb{Q}_2)$$

which acts on the apartment A as a reflection with respect to the vertex $v_{\frac{1}{2}}$. Indeed, we have:

$$\begin{aligned}
RL_{\frac{1}{2}} &= \langle 2Re_1,\ Re_2,\ Re_3,\ Re_4,\ Re_5 \rangle = \langle e_5,\ e_2,\ e_3,\ e_4,\ 2e_1 \rangle = L_{\frac{1}{2}},\\
RL_0 &= \langle Re_1,\ Re_2,\ Re_3,\ Re_4,\ Re_5 \rangle = \left\langle \frac{1}{2}e_5,\ e_2,\ e_3,\ e_4,\ 2e_1 \right\rangle = L_1,\\
RL_1 &= \left\langle 2Re_1,\ Re_2,\ Re_3,\ Re_4,\ \frac{1}{2}Re_5 \right\rangle = \langle e_5,\ e_2,\ e_3,\ e_4,\ e_1 \rangle = L_0.
\end{aligned}$$

We can use the same observations as before and the reflection R to say that every neighbor of $v_{\frac{1}{2}}$ is in the same orbit of v_0 under the action of the stabilizer in $\mathrm{O}(q;\mathbb{Q}_2)$ of $v_{\frac{1}{2}}$ (which is the same as the stabilizer of $L_{\frac{1}{2}}$).

We now consider $L_{\frac{1}{2}}$ and $2L_0$ as representatives of $v_{\frac{1}{2}}$ and v_0. Having

$$2L_{\frac{1}{2}} \subsetneq 2L_0 \subsetneq L_{\frac{1}{2}} = \langle f_1, \dots, f_5 \rangle\,,$$

it follows that the vector space $V_{\frac{1}{2}} = L_{\frac{1}{2}}/2L_{\frac{1}{2}}$ with basis $\{\overline{f_i}\}_{i\leq 5}$ contains the image of $2L_0$ as a subspace of dimension 1 spanned by the vector $\overline{f_1}$. We notice that the quadratic form $\overline{q}_{\frac{1}{2}}$ on $V_{\frac{1}{2}}$ induced by the restriction of q to $L_{\frac{1}{2}}$ is degenerate, and its radical (i.e. its totally degenerate maximal subspace) is the subspace $\langle \overline{f_1}, \overline{f_5} \rangle$.

This implies that each orthogonal transformation of the quadratic form $\overline{q}_{\frac{1}{2}}$ must preserve this subspace, and therefore the orbit of the line spanned by $\overline{f_1}$ consists of at most three lines: $\langle \overline{f_1} \rangle, \langle \overline{f_5} \rangle$ and $\langle \overline{f_1} + \overline{f_5} \rangle$. Each of these lines are actually in the orbit we are studying. Indeed, the element $\langle \overline{f_1} \rangle$ identifies the lattice $2L_0$ and hence the vertex v_0; the element $\langle \overline{f_5} \rangle$ is the projection to the quotient of the lattice $2L_1$, which gives us the other already known neighbor of $v_{\frac{1}{2}}$, that is v_1; finally, the lattice

$$L = \langle e_1 + \frac{1}{2}e_5,\ e_2,\ e_3,\ e_4,\ e_5 \rangle$$

is such that $2L_{\frac{1}{2}} \subsetneq 2L \subsetneq L_{\frac{1}{2}}$, the projection of $2L$ is the line $\langle \overline{f_1} + \overline{f_5} \rangle$ and its homothety class is a vertex adjacent to $v_{\frac{1}{2}}$ since the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -1 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(q;\mathbb{Q}_2)$$

exchanges v_0 and $[L]$ stabilizing $v_{\frac{1}{2}}$.

It is then proved that $v_{\frac{1}{2}}$ has valence 3. □

Having finally completed this parenthesis about the structure of the Bruhat-Tits tree $\mathcal{T}$ of the group $\mathrm{O}(q;\mathbb{Q}_2)$, we are now ready to repeat the construction done in the previous section for the simpler example of the magic manifold, in order to produce a congruence RFRS tower for the finite index subgroup $\Gamma(4)$.

Proposition 3.16. *The congruence subgroup* $\Gamma(4)$ *of level 4 in* $\mathrm{O}(4,1;\mathbb{Z})$ *admits a congruence RFRS tower. In particular, it admits a RFRS tower* $\{\Gamma_n\}_{n\in\mathbb{N}}$ *with the property that for every* n *there exists a* j_n *such that* $\Gamma(2^{j_n}) \leq \Gamma_n$.

Proof. As already said, $\Gamma(4)^{ab}$ has no 2-torsion, hence our purpose is to apply Lemma 3.7 with respect to the finite index subgroup $\Gamma(4)$ of $\mathrm{O}(q;\mathbb{Q}_2)$. Recall from Proposition 2.33 that every element in $\mathrm{O}(q;\mathbb{Q})$ commensurates $\Gamma(4)$. We will choose inside there the elements g_n.

Consider the matrix

$$g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{SO}(q;\mathbb{Q}) < \mathrm{SO}(q;\mathbb{Q}_2)$$

that exchanges the vertices v_0 and v_1 in $\mathcal{T}$ and fixes the intermediate vertex $v_{\frac{1}{2}}$.

We now set $\Gamma_0 = \Gamma(4)$ and $\Delta_1 = g_1\Gamma_0 g_1^{-1}$. As before, we claim that

$$\Gamma(16) \leq \Gamma_0 \cap g_1\Gamma_0 g_1^{-1}.$$

To see this we just have to do some easy calculations:

$$g_1 \begin{pmatrix} 1+16e_5 & -32e_2 & -32e_3 & -32e_4 & 64e_1 \\ -8b_5 & 1+16b_2 & 16b_3 & 16b_4 & -32b_1 \\ -8c_5 & 16c_2 & 1+16c_3 & 16c_4 & -32c_1 \\ -8d_5 & 16d_2 & 16d_3 & 1+16d_4 & -32d_1 \\ 4a_5 & -8a_2 & -8a_3 & -8a_4 & 1+16a_1 \end{pmatrix} g_1^{-1} =$$

$$= \begin{pmatrix} 1+16a_1 & 16a_2 & 16a_3 & 16a_4 & 16a_5 \\ 16b_1 & 1+16b_2 & 16b_3 & 16b_4 & 16b_5 \\ 16c_1 & 16c_2 & 1+16c_3 & 16c_4 & 16c_5 \\ 16d_1 & 16d_2 & 16d_3 & 1+16d_4 & 16d_5 \\ 16e_1 & 16e_2 & 16e_3 & 16e_4 & 1+16e_5 \end{pmatrix}$$

where a_i, b_i, c_i, d_i, e_i are integers for every $1 \leq i \leq 5$. Since $g_1 \in \mathrm{SO}(q;\mathbb{Q})$, the matrix being conjugated on the left-hand side preserves the quadratic form q if and only if the matrix on the right-hand side does. This proves the claim.

Therefore, by applying Lemma 3.8 we find that $\Gamma_0/\left(\Gamma_0 \cap g_1\Gamma_0 g_1^{-1}\right)$ is an abelian 2-group.

We now define g_n inductively as follows. Let v_n be a vertex of $\mathcal{T}$ in the $\mathrm{O}(q;\mathbb{Q}_2)$-orbit of v_0 that is at distance 2 in $\mathcal{T}$ from some vertex v_i in $\{v_0, \ldots, v_{n-1}\}$. Since $\mathrm{O}(q;\mathbb{Q})$ is dense in $\mathrm{O}(q;\mathbb{Q}_2)$, we can choose a $h_n \in \mathrm{O}(q;\mathbb{Q})$ such that $h_n \cdot v_0 = v_i$ and $h_n \cdot v_1 = v_n$. Define then $g_n := h_n g_1 h_n^{-1}$ and $\Delta_n := g_n \Delta_i g_n^{-1}$.

Our next claim is that $\Delta_i/(\Delta_i \cap \Delta_n)$ is an abelian 2-group, and the proof is exactly the same as for the magic manifold: by conjugating the quotient with h_n^{-1} we obtain that

$$h_n^{-1}(\Delta_i/(\Delta_i \cap \Delta_n))h_n \cong \Delta_0/(\Delta_0 \cap \Delta_1),$$

and consequently that $\Delta_i/(\Delta_i \cap \Delta_n)$ is an abelian 2-group.

By choosing the sequence $\{v_n\}_{n\in\mathbb{N}}$ to exhaust the $\mathrm{O}(q;\mathbb{Q}_2)$-orbit of v_0, and defining

$$\Delta_n = g_n\Gamma_0 g_n^{-1}, \qquad \Gamma_n = \bigcap_{i=0}^{n} \Delta_i$$

as in Lemma 3.7, we have that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is cofinal and $\{g_n\}_{n\in\mathbb{N}}$ satisfies the required assumptions by construction. Indeed, each Δ_n fixes the vertex v_n and the intersection

$$\bigcap_{n\in\mathbb{N}} \Gamma_n$$

fixes every such vertex; but then, each other vertex in $\mathcal{T}$, which belongs to the $\mathrm{O}(q;\mathbb{Q}_2)$-orbit of $v_{\frac{1}{2}}$, lies between two fixed vertices and thus it is necessarily stabilized too.

Therefore we have defined a RFRS tower for $\Gamma(4)$.

It only remains to check that every Γ_n contains a principal congruence subgroup of level 2^{j_n} for some j_n. Again, the same principle as in the previous example holds: each Γ_n is the intersection of conjugates of $\Delta_0 = \Gamma(4)$, and as before it is possible to prove [Sto20] that each intersection $\Delta_0 \cap \Delta_n$ contains a deep enough principal congruence subgroup of level 2^{k_n}. From this it follows that, if we set j_n to be

$$j_n := \max\{k_1 = 4, k_2, \ldots, k_n\},$$

then we have

$$\Gamma(2^{j_n}) \leq (\Delta_0 \cap \Delta_1) \cap (\Delta_0 \cap \Delta_2) \cap \ldots \cap (\Delta_0 \cap \Delta_n) = \Delta_0 \cap \ldots \cap \Delta_n = \Gamma_n.$$

This completes the proof. $\square$

We only talked about the group $\mathrm{O}(4,1;\mathbb{Z})$ at the moment, but we recall that the main result of this chapter, Theorem 3.2, concerns RFRS towers inside $\mathrm{PSL}(2,\mathcal{O}_d)$. To obtain this, we need to find some commensurability argument.

Let us state the following classic result of number theory: the constraint on d in Theorem 3.2 comes from here.

Theorem 3.17 (Gauss-Legendre, [Dir50])**.** *Let d a positive integer. Then d is the sum of three squares if and only if d is not of the form $4^t(8k-1)$.*

Using this fact, if we define the quadratic form $q_d(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 - dx_4^2$ for d a positive square-free number then it follows that q_d is *isotropic* (i.e. there exists a non-zero vector y such that $q_d(y) = 0$) if and only if $d \not\equiv -1 \pmod 8$.

This isotropy is fundamental in finding a connection between the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_d)$ and $O(4, 1; \mathbb{Z})$. Indeed, recall from Chapter 1 that we can identify the connected component of the identity $\mathrm{SO}^+(3, 1; \mathbb{R})$ inside $\mathrm{O}(3, 1; \mathbb{R})$ with $\mathrm{Isom}^+(\mathbb{H}^3)$, and we can deduce from [BHC62] that $\mathrm{SO}^+(q_d; \mathbb{Z})$ has finite covolume (when considered up to comensurability inside $\mathrm{SO}^+(3, 1; \mathbb{R})$). The fact that q_d is isotropic then implies, again from [BHC62], that $\mathrm{SO}^+(q_d; \mathbb{Z})$ is not cocompact.

The following theorem enlightens the first step about the connection we are chasing:

Theorem 3.18. *Let $d \not\equiv -1 \pmod 8$ be a positive square-free number and let q_d be the quadratic form defined above. Then the group $\mathrm{SO}^+(q_d; \mathbb{Z})$ is commensurable, when passing through the isomorphisms $\mathrm{SO}^+(3, 1; \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$, with the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_d)$.*

Moreover, the subgroup $\Delta \leq \mathrm{PSL}(2, \mathcal{O}_d)$ being embedded in $\mathrm{SO}^+(q_d; \mathbb{Z})$ is a congruence subgroup of both and the pre-image of every congruence subgroup with respect to the embedding is a congruence subgroup.

We postpone this proof since we will need some arguments which are easier to present in other settings, and that we are going to see in the next few proofs.

With the previous theorem in mind, the next step to do is to add a dimension. We consider a new quadratic form on 5 variables and we see q_d and q_0 as part of it.

Lemma 3.19. *The quadratic form $p_d = \langle d \rangle + q_d$, that is to say the quadratic form*

$$p_d(x_1, x_2, x_3, x_4, x_5) = dx_1^2 + x_2^2 + x_3^2 + x_4^2 - dx_5^2,$$

is equivalent over $\mathbb{Q}$ to the quadratic form q_0 defined at the beginning of this section.

Proof. Let $P_d = \mathrm{diag}(d, 1, 1, 1, -d)$ be the matrix representing the quadratic form p_d and let Q_0 be the matrix representing q_0 as defined at the beginning of this section. The matrix $M \in \mathrm{GL}(5, \mathbb{Q})$ which realizes the equivalence is

$$M = \begin{pmatrix} \frac{1+d}{2d} & 0 & 0 & 0 & \frac{1-d}{2d} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1-d}{2d} & 0 & 0 & 0 & \frac{1+d}{2d} \end{pmatrix}.$$

Indeed, an immediate calculation shows that $M^T P_d M = Q_0$ and the claim follows. $\square$

The equivalence of quadratic forms gives us also that:

Lemma 3.20. *The group* $\mathrm{O}(4,1;\mathbb{Q})$ *is the conjugate of* $\mathrm{O}(p_d;\mathbb{Q})$ *with respect to the matrix* M *of the previous lemma, that is to say we have*

$$\mathrm{O}(4,1;\mathbb{Q}) = M^{-1}\,\mathrm{O}(p_d;\mathbb{Q})M,$$

while the subgroup of $\mathbb{Z}$*-points* $\mathrm{O}(p_d;\mathbb{Z})$ *is conjugated to a subgroup of* $\mathrm{O}(4,1;\mathbb{Q})$ *commensurable with* $\mathrm{O}(4,1;\mathbb{Z})$*.*

Moreover, the intersection $M^{-1}\,\mathrm{O}(p_d;\mathbb{Z})M \cap \mathrm{O}(4,1;\mathbb{Z})$ *is a congruence subgroup of* $\mathrm{O}(4,1;\mathbb{Z})$ *and its conjugate* $\mathrm{O}(p_d;\mathbb{Z}) \cap M\,\mathrm{O}(4,1;\mathbb{Z})M^{-1}$ *is a congruence subgroup of* $\mathrm{O}(p_d;\mathbb{Z})$*. Using a little abuse of notations we say that the common subgroup of* $\mathrm{O}(p_d;\mathbb{Z})$ *and* $\mathrm{O}(4,1;\mathbb{Z})$ *is a congruence subgroup of both.*

Proof. The equality is proved by a simple calculus: let A be an element in $\mathrm{O}(p_d;\mathbb{Q})$, then

$$\begin{aligned}(M^{-1}AM)^T Q_0 (M^{-1}AM) &= M^T A^T (M^{-1})^T Q_0 M^{-1} AM = \\ &= M^T A^T P_d A M = M^T P_d M = Q_0.\end{aligned}$$

This implies the inclusion $\mathrm{O}(4,1;\mathbb{Q}) \supseteq M^{-1}\,\mathrm{O}(p_d;\mathbb{Q})M$. The other is done similarly.

To obtain the result about $\mathbb{Z}$-points, let m be the least common multiple of all the denominators of the entries of M and M^{-1}. Then $M^{-1}\Gamma_{p_d}(m^2)M$, where $\Gamma_{p_d}(m^2)$ is the principal congruence subgroup of level m^2 inside $\mathrm{O}(p_d;\mathbb{Z})$, is a subgroup of $\mathrm{O}(4,1;\mathbb{Z})$ since if we write any element in $\Gamma_{p_d}(m^2)$ as $I + m^2 N$ it follows that

$$M^{-1}(I + m^2 N)M = I + (mM^{-1})N(mM)$$

is a sum of products of integral matrices. It preserves the quadratic form q_0 by the previous argument. Now, the subgroup $M^{-1}\Gamma_{p_d}(m^2)M$ is obviously of finite index inside $M^{-1}\,\mathrm{O}(p_d;\mathbb{Z})M$, and to see that it is also of finite index inside $\mathrm{O}(4,1;\mathbb{Z})$ it is sufficient to show that it is a congruence subgroup. Indeed, let $\Gamma_{q_0}(m^4)$ be the principal congruence subgroup of level m^4 inside $\mathrm{O}(4,1;\mathbb{Z})$. Then, writing any element there as $I + m^4 N$, we have that

$$M(I + m^4 N)M^{-1} = I + m^2(mM)N(mM^{-1})$$

which proves the inclusion $M\Gamma_{q_0}(m^4)M^{-1} \leq \Gamma_{p_d}(m^2)$ and hence the commensurability.

Note that we also implicitly proved the last statement about the common subgroup being a congruence subgroup of both, since

$$\Gamma_{q_0}(m^4) \leq M^{-1}\Gamma_{p_d}(m^2)M \leq M^{-1}\,\mathrm{O}(p_d;\mathbb{Z})M \cap \mathrm{O}(4,1;\mathbb{Z}).$$

□

Corollary 3.21. *There is a natural embedding* $\mathrm{O}(q_d;\mathbb{Z}) \hookrightarrow \mathrm{O}(4,1;\mathbb{Z})$ *up to commensurability. In particular, this embedding preserves congruence subgroups in both directions, that is to say both the image and the pre-image of a congruence subgroup is again a congruence subgroup.*

Proof. The embedding is the composition of the map $\mathrm{O}(q_d;\mathbb{Z}) \hookrightarrow \mathrm{O}(p_d;\mathbb{Z})$ given by $A \longmapsto (1)\oplus A$, where

$$(1)\oplus A = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right),$$

and the commensurability argument of Lemma 3.20 for $\mathrm{O}(p_d;\mathbb{Z})$ and $\mathrm{O}(4,1;\mathbb{Z})$ given by the equivalence of the two quadratic forms.

The first map preserves the congruence subgroups trivially. From what we proved in the previous lemma we also have that the subgroup $G \leq \mathrm{O}(p_d;\mathbb{Z})$ being embedded in $\mathrm{O}(4,1;\mathbb{Z})$ is a congruence subgroup of both. We now see that this suffice to prove the thesis.

The fundamental observation is that the intersection of congruence subgroups is again a congruence subgroup: this is easy to see in greater generality, since if H_1 and H_2 are congruence subgroups inside $\mathrm{GL}(n,\mathbb{Z})$ then there exist m_1 and m_2 in $\mathbb{Z}$ such that

$$\Gamma(m_i) \leq H_i$$

for $i = 1, 2$, and it is straightforward that $\Gamma(m_1 m_2) \leq H_1 \cap H_2$.

Returning to our setting, let $H \leq \mathrm{O}(p_d;\mathbb{Z})$ be any congruence subgroup. Then, by intersecting it with G we have that $H \cap G$ is a congruence subgroup for what we just said. Finally, by embedding it inside $\mathrm{O}(4,1;\mathbb{Z})$ we conclude that it also defines a congruence subgroup of $\mathrm{O}(4,1;\mathbb{Z})$: there exists an m' such that $\Gamma(m') \leq H$, and the very same calculations as in Lemma 3.20 lead to

$$\Gamma_{q_0}(m^4 m') \leq M^{-1}\Gamma_{p_d}(m^2 m')M \leq M^{-1}(H \cap G)M.$$

Viceversa, with the same calculations we can prove that if $H \leq \mathrm{O}(4,1;\mathbb{Z})$ is a congruence subgroup, then the intersection $MHM^{-1} \cap G$ of its conjugate MHM^{-1} with $\mathrm{O}(p_d;\mathbb{Z})$ is a congruence subgroup of $\mathrm{O}(p_d;\mathbb{Z})$.

Hence congruence subgroups are preserved by passing (up to commensurability) from $\mathrm{O}(q_d;\mathbb{Z})$ to $\mathrm{O}(4,1;\mathbb{Z})$ and viceversa. $\square$

We are now ready to present the proof of Theorem 3.18:

Proof of Theorem 3.18. We distinguish two cases: the first one is $d \equiv 1, 2 \pmod 4$, while the second is $d \equiv 3 \pmod 8$. Depending on the case, we will embed the whole group $\mathrm{PSL}(2,\mathcal{O}_d)$ in the group of transformations of a particular quadratic space, different from $\mathrm{SO}^+(q_d;\mathbb{Z})$, but then we will pass up to commensurability to $\mathrm{SO}^+(q_d;\mathbb{Z})$ using almost the same arguments as in Lemma 3.20.

Let us analyze the two cases separately: if $d \equiv 1, 2 \pmod 4$ we define the quadratic form

$$q'_d(x_1,x_2,x_3,x_4) := 2x_1x_2 + 2x_3^2 + 2dx_4^2$$

and the homomorphism $\varphi_d\colon \mathrm{PSL}(2,\mathcal{O}_d) \longrightarrow \mathrm{SO}^+(q_d';\mathbb{Z})$ by sending a matrix

$$\begin{pmatrix} a_0 + a_1\sqrt{-d} & b_0 + b_1\sqrt{-d} \\ c_0 + c_1\sqrt{-d} & d_0 + d_1\sqrt{-d} \end{pmatrix}$$

to the matrix

$$\begin{pmatrix} d_0^2 + dd_1^2 & -c_0^2 - dc_1^2 & 2(c_0d_0 + dc_1d_1) & -2d(c_1d_0 - c_0d_1) \\ -b_0^2 - db_1^2 & a_0^2 + da_1^2 & -2(a_0b_0 + da_1b_1) & 2d(a_1b_0 - a_0b_1) \\ b_0d_0 + db_1d_1 & -a_0c_0 - da_1c_1 & b_0c_0 + db_1c_1 + a_0d_0 + da_1d_1 & d(b_1c_0 - b_0c_1 - a_1d_0 + a_0d_1) \\ b_1d_0 - b_0d_1 & a_0c_1 - a_1c_0 & b_1c_0 - b_0c_1 + a_1d_0 - a_0d_1 & -b_0c_0 - db_1c_1 + a_0d_0 + da_1d_1 \end{pmatrix}.$$

It can be seen in [Chu19] or [JM96] that this is a well-defined map and that it is injective. Moreover, it is easy to check from the explicit definition given above that if $m \in \mathbb{Z}$ then the congruence subgroup $\Gamma(m) \leq \mathrm{PSL}(2,\mathcal{O}_d)$ of level m is mapped into the congruence subgroup of level m of the image $\varphi_d(\mathrm{PSL}(2,\mathcal{O}_d))$. This proves that the pre-image of any congruence subgroup of $\mathrm{SO}^+(q_d';\mathbb{Z})$ contains the principal congruence subgroup of the same level in $\mathrm{PSL}(2,\mathcal{O}_d)$.

Now, we have to prove that we can pass up to commensurability from $\mathrm{SO}^+(q_d';\mathbb{Z})$ to $\mathrm{SO}^+(q_d;\mathbb{Z})$ and that the common subgroup can be taken to be a congruence subgroup of both. Indeed, by a change of basis we have that q_d' is equivalent over $\mathbb{Q}$ to the diagonal quadratic form $\langle -2, 2, 2, 2d\rangle = 2 \cdot \langle -1, 1, 1, d\rangle$:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2d \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2d \end{pmatrix}.$$

Unfortunately, by computing some invariants which characterize $\mathbb{Q}$-equivalent quadratic forms (see for instance [Ser73, Chapter 4]) it can be shown that the forms q_d' and q_d are not equivalent.

Hence we need to relax the definition of equivalence, by replacing it with the notion of *projective equivalence*, that is to say being equivalent over $\mathbb{Q}$ up to a rational non-zero scalar. This slight modification doesn't ruin the arguments of Lemma 3.20, which can be generalized with the same thesis and the same proof if the two quadratic forms we are considering are projectively equivalent. Indeed, if two forms f and g are equivalent under this new notion (and they are represented by matrices F and G), then there exist an $a \in \mathbb{Q}^*$ and a matrix $M \in \mathrm{GL}(n,\mathbb{Q})$ such that $M^TFM = aG$, and since the groups $\mathrm{O}(ag;\mathbb{Z})$ and $\mathrm{O}(g;\mathbb{Z})$ coincide everything repeats verbatim.

It can be proved now that q_d' and q_d are projectively equivalent using [MA13, Theorem 8] and the complete set of invariants listed in that article, and this ensures the commensurability of $\mathrm{SO}^+(q'_d; 1z)$ and $\mathrm{SO}^+(q_d; \mathbb{Z})$ together with the existence of a common congruence subgroup. The same arguments as those used in the proof of Corollary 3.21 then prove that passing through this commensurability preserves the congruence subgroups in both directions.

Let us now analyze the case $d \equiv 3 \pmod 8$: under this assumption we define the quadratic form

$$q'_d(x_1, x_2, x_3, x_4) := 2x_1x_2 + 2x_3^2 + 2x_3x_4 + \frac{d+1}{2}x_4^2$$

and the homomorphism $\varphi_d\colon \mathrm{PSL}(2, \mathcal{O}_d) \longrightarrow \mathrm{SO}^+(q'_d; \mathbb{Z})$ by sending a matrix

$$\begin{pmatrix} a_0 + a_1\frac{1+\sqrt{-d}}{2} & b_0 + b_1\frac{1+\sqrt{-d}}{2} \\ c_0 + c_1\frac{1+\sqrt{-d}}{2} & d_0 + d_1\frac{1+\sqrt{-d}}{2} \end{pmatrix}$$

to the following matrix, where we write $d = 4k - 1$:

$$\left(\begin{matrix} d_0^2 + d_1d_0 + kd_1^2 & -c_0^2 - c_1c_0 - kc_1^2 & 2c_0d_0 + c_1d_0 + c_0d_1 + 2kc_1d_1 \\ -b_0^2 - b_1b_0 - kb_1^2 & a_0^2 + a_1a_0 + ka_1^2 & -2a_0b_0 - a_1b_0 - a_0b_1 - 2ka_1b_1 \\ b_0d_0 + b_1d_0 + kb_1d_1 & -a_0c_0 - a_1c_0 - ka_1c_1 & b_0c_0 + b_1c_0 + kb_1c_1 + a_0d_0 + a_1d_0 + ka_1d_1 \\ b_0d_1 - b_1d_0 & a_1c_0 - a_0c_1 & -b_1c_0 + b_0c_1 - a_1d_0 + a_0d_1 \end{matrix}\right.$$
$$\left.\begin{matrix} c_0d_0 + 2kc_1d_0 - 2kc_0d_1 + c_0d_1 + kc_1d_1 \\ -a_0b_0 - 2ka_1b_0 + 2ka_0b_1 - a_0b_1 - ka_1b_1 \\ b_0c_0 - kb_1c_0 + b_1c_0 + kb_0c_1 + kb_1c_1 + ka_1d_0 - ka_0d_1 \\ -b_0c_0 - b_1c_0 - kb_1c_1 + a_0d_0 + a_0d_1 + ka_1d_1 \end{matrix}\right).$$

As before, it can be seen (from [Chu19] or [JM96]) that this is a well-defined injective homomorphism, and that every level-m congruence subgroup with $m \in \mathbb{Z}$ is mapped into the congruence subgroup of level m of the image $\varphi_d(\mathrm{PSL}(2, \mathcal{O}_d))$. Hence the thesis about the pre-image of every congruence subgroup being congruence is satisfied in this case too.

By a rational change of basis we can transform the quadratic form q'_d to the same diagonal quadratic form $\langle -2, 2, 2, 2d\rangle = 2 \cdot \langle -1, 1, 1, d\rangle$ as in the previous case:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & \frac{d+1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2d \end{pmatrix}.$$

Hence, by using the same result as before [MA13, Theorem 8] we can conclude that q'_d is projectively equivalent to q_d and that $\mathrm{SO}^+(q'_d; \mathbb{Z})$ is commensurable with $\mathrm{SO}^+(q_d; \mathbb{Z})$ having a common congruence subgroup. Using again the arguments of Corollary 3.21 we have that the congruence property is preserved by this commensurability passage. This concludes the proof.

We stress that we could also define the quadratic form

$$q_d'(x_1, x_2, x_3, x_4) = 2x_1x_2 + 2x_3^2 + 2x_3x_4 + \frac{d+1}{2}x_4^2$$

when $d \equiv 7 \pmod 8$, and the homomorphism $\varphi_d \colon \mathrm{PSL}(2, \mathcal{O}_d) \longrightarrow \mathrm{SO}^+(q_d'; \mathbb{Z})$ would be well-defined and injective. But the quadratic forms q_d and q_d' are not projectively equivalent in this case as we can see in [MA13, Corollary 4], and hence we would not have the commensurability between $\mathrm{PSL}(2, \mathcal{O}_d)$ and $\mathrm{SO}^+(q_d; \mathbb{Z})$. $\square$

We are now ready to complete the proof of the main theorem of this Chapter, that is to say Theorem 3.2. But before doing this, we briefly recap the passages we followed in order to embed (up to commensurability) the group $\mathrm{PSL}(2, \mathcal{O}_d)$ inside $\mathrm{SO}(4, 1; \mathbb{Z})$:

1. We defined an embedding $\mathrm{PSL}(2, \mathcal{O}_d) \hookrightarrow \mathrm{SO}^+(q_d'; \mathbb{Z})$ of the Bianchi groups inside the group of transformations of a 4-dimensional bilinear space in such a way that the pre-image of every congruence subgroup is still a congruence subgroup;

2. We defined a map $\mathrm{SO}^+(q_d'; \mathbb{Z}) \dashrightarrow \mathrm{SO}^+(q_d; \mathbb{Z})$ which is defined only on a congruence subgroup of $\mathrm{SO}^+(q_d'; \mathbb{Z})$ and which maps it isomorphically onto a congruence subgroup of $\mathrm{SO}^+(q_d; \mathbb{Z})$. Moreover, we proved that this change of perspective doesn't ruin the property of a subgroup being congruence;

3. We defined an embedding $\mathrm{SO}^+(q_d; \mathbb{Z}) \hookrightarrow \mathrm{SO}^+(p_d; \mathbb{Z})$ of the group $\mathrm{SO}^+(q_d; \mathbb{Z})$ inside the group of transformations on the integers which preserve the quadratic form $p_d = \langle d \rangle + q_d$. Moreover, this embedding preserves the congruence subgroups in both directions;

4. We defined a map $\mathrm{SO}^+(p_d; \mathbb{Z}) \dashrightarrow \mathrm{SO}^+(4, 1; \mathbb{Z})$ which is defined only on a congruence subgroup of $\mathrm{SO}^+(p_d; \mathbb{Z})$ and which maps it isomorphically onto a congruence subgroup of $\mathrm{SO}^+(4, 1; \mathbb{Z})$. Moreover, we proved that this isomorphism preserves the congruence subgroups.

We state again the main theorem for a better readability:

Theorem 3.2. *The Bianchi groups* $\mathrm{PSL}(2, \mathcal{O}_d)$ *with* $d \not\equiv -1 \pmod 8$ *and d positive and square-free contain a RFRS tower consisting entirely of congruence subgroups. In particular, these Bianchi orbifolds virtually fiber on a congruence cover.*

Proof. By putting together what we proved we have that there exists a congruence subgroup $\Delta \leq \mathrm{PSL}(2, \mathcal{O}_d)$ which embeds into $\mathrm{SO}(4, 1; \mathbb{Z})$ and such that its image (which we still denote Δ with a little abuse of notation) is also a congruence subgroup. In Proposition 3.16 we constructed a RFRS tower $\{\Gamma_n\}_{n \in \mathbb{N}}$ for the congruence subgroup $\Gamma(4) \leq \mathrm{O}(4, 1; \mathbb{Z})$ which consists of congruence subgroups. Hence, by intersecting them with Δ we obtain a RFRS tower (we recall that the RFRS property passes to subgroups) for Δ and the subgroups $\Delta \cap \Gamma_n$ are congruence subgroups being the intersection of congruence subgroups.

By taking the pre-image and looking at these subgroups in $\mathrm{PSL}(2, \mathcal{O}_d)$ we obtain a RFRS tower for Δ consisting entirely of congruence subgroups, since we have proved that the pre-image of any congruence subgroup is still a congruence subgroup. $\square$

We conclude this thesis with a brief recap of the construction we introduced, which obviously works in greater generality than the two examples we presented here.

If k is a number field and $\mathcal{G}_{\mathcal{O}_k}$ is an arithmetic lattice in $\mathrm{SO}(n, 1; \mathbb{R})$, that is to say a finite covolume arithmetic subgroup defined as the set of the $\mathcal{O}_k$-points in an algebraic group $\mathcal{G}$ defined over k, then we can try to repeat the method of constructing a RFRS tower for a finite index subgroup of $\mathcal{G}_{\mathcal{O}_k}$ we used in our examples.

Indeed, if $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_k$ with residue characteristic p, and $\Gamma(\mathfrak{p})$ denotes the principal congruence subgroup of $\mathcal{G}_{\mathcal{O}_k}$ of level $\mathfrak{p}$, we can apply our inductive method to any finite index subgroup $\Gamma \leq \Gamma(\mathfrak{p})$ such that $H_1(\Gamma; \mathbb{Z})$ has no p-torsion.

Explicitly, if we consider the completion $k_{\mathfrak{p}}$ of the number field k with respect to the $\mathfrak{p}$-adic valuation, we can use the Bruhat-Tits building for $\mathcal{G}_{k_{\mathfrak{p}}}$ as an organizational tool in order to define a sequence of elements $\{g_n\} \subseteq \mathcal{G}_k$ which commensurate Γ and such that

$$\left\{ \bigcap_{j=0}^{n} g_j \Gamma g_j^{-1} \right\}_{n \in \mathbb{N}}$$

is a RFRS tower for Γ. The key is to find an initial $g_1 \in \mathcal{G}_k$ so that

$$\Gamma / (\Gamma \cap g_1 \Gamma g_1^{-1})$$

is an elementary abelian p-group. Then, every other g_n is defined inductively thanks to the structure of the Bruhat-Tits building associated to $\mathcal{G}_{k_{\mathfrak{p}}}$. Moreover, again by the structure of the Bruhat-Tits building, it can be proved that every element of the RFRS tower defined is a congruence subgroup of $\mathcal{G}_{\mathcal{O}_k}$.

As a final remark, notice that if the congruence subgroup of level 4 in $\mathrm{SO}(6, 1; \mathbb{Z})$ has no 2-torsion in its first homology (i.e. its abelianization), then Theorem 3.2 holds for all Bianchi groups. This is because every Bianchi group embeds up to commensurability in $\mathrm{O}(6, 1; \mathbb{Z})$ and the common subgroup is a congruence subgroup of both (see [Chu19] or [ALR01]). More generally, one only needs to find a prime p so that the congruence subgroup of level p in $\mathrm{SO}(6, 1; \mathbb{Z})$ has no p-torsion in its abelianization, which seems likely but very difficult to verify computationally. If this holds, then all Bianchi groups would contain a congruence RFRS tower and hence they would fiber on a congruence cover.

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